MATH 556 - PRACTICE EXAM QUESTIONS II SOLUTIONS

1. (a) Using properties of cfs, we have

$$C_{T_n}(t) = \left\{ e^{-|t|} \right\}^n = e^{-|nt|}$$

Now using the scale transformation result for mgfs/cfs (given on Formula Sheet), we have that if $V = \lambda U$, then

$$C_V(t) = C_U(\lambda t)$$

we deduce that, in distribution, $T_n = nX$, where $X \sim Cauchy$, so that, by the univariate transformation theorem,

$$f_{T_n}(x) = f_X(x/n)|J(x)| = \frac{1}{\pi}\frac{1}{1+(x/n)^2}\frac{1}{n} = \frac{1}{\pi}\frac{n}{n^2+x^2}$$

(b) From (a), we can deduce immediately that $\overline{X}_n \sim Cauchy$ **for all** n**.** Hence, using the Cauchy cdf,

$$P[|\overline{X}_n| > \epsilon] = 1 - \frac{2}{\pi}\arctan(\epsilon) \nrightarrow 0 \qquad \text{as} \qquad n \longrightarrow \infty.$$

and hence $\overline{X}_n \overset{p}{\nrightarrow} 0$ as $n \longrightarrow \infty$.

(c) Many possible methods of solution; recall that the scale mixture formulation specifies a three level hierarchy in this case

$$\begin{array}{lll} \text{LEVEL 3 :} & \alpha, \beta > 0 & \text{Fixed parameters} \\ \text{LEVEL 2 :} & V \sim Gamma(\alpha, \beta) & \\ \text{LEVEL 1 :} & X|V = v \sim Normal(0, g(v)) & \end{array}$$

for some non-negative function $g(.)$. The marginal for X is thus

$$f_X(x) = \int_0^\infty f_{X|V}(x|v) f_V(v)\,dv = \int_0^\infty \left(\frac{1}{2\pi g(v)}\right)^{1/2} \exp\left\{-\frac{x^2}{2g(v)}\right\} \frac{\beta^\alpha}{\Gamma(\alpha)} v^{\alpha-1} e^{-\beta v}\,dv.$$

We require the result of this calculation to be the Cauchy pdf. In order to integrate out v, it appears that we must make the integrand proportional to a Gamma pdf, and choosing $g(v) = v^{-1}$ makes this possible; ignoring constants, the integrand becomes

$$v^{\alpha+1/2-1} \exp\left\{-\frac{v(2\beta+x^2)}{2}\right\}$$

which, on integration, yields a term proportional to

$$\frac{\Gamma(\alpha+1/2)}{(2\beta+x^2)^{\alpha+1/2}}.$$

Hence choosing $\alpha = 1/2$, $\beta = 1/2$ yields a term proportional to the Cauchy pdf. Thus the Cauchy distribution is a scale mixture of a Normal density by a $Gamma(1/2, 1/2) \equiv \chi^2_1$ distribution, with "link" function $g(v) = v^{-1}$.

2. (a) Given a f_X, we construct a tilted version with tilt given parameter θ as follows; we consider

$$f_{X|\theta}(x|\theta) \propto f_X(x)\exp\{\theta X\}$$

in such a way so that the resulting function $f_{X|\theta}(x|\theta)$ is a valid pdf. Clearly this function is non-negative, and integrable if

$$\int_{-\infty}^{\infty} f_X(x)\exp\{\theta x\}\ dx < \infty$$

If this holds, then

$$f_{X|\theta}(x|\theta) = \frac{f_X(x)\exp\{\theta X\}}{\displaystyle\int_{-\infty}^{\infty} f_X(x)\exp\{\theta x\}\ dx} = \frac{f_X(x)\exp\{\theta X\}}{M_X(\theta)}$$

where M_X is the mgf corresponding to the original f_X. Finally, if $K_X(t) = \log M_X(t)$ is the corresponding cumulant generating function, then

$$f_{X|\theta}(x|\theta) = f_X(x)\exp\{\theta X - K_X(\theta)\}$$

This is a natural exponential family distribution in its canonical parameterization, that is,

$$f_{X|\theta}(x|\theta) = h(x)c(\theta)\exp\{\theta X\}$$

where $h(x) = f_X(x)$ and $c(\theta) = M_X(\theta)$. This computation can be generalized by considering the derivation with random variable $S = s(X)$ replacing X in the exponent, and M_S replacing M_X.

(b) If $\mathcal{N}$ is given by

$$\mathcal{N} \equiv \left\{\theta \in \mathbb{R} : K_S(\theta) = \log\left[\int e^{s(y)\theta} f_Y(y)\ dy\right] < \infty\right\}$$

(i) $0 \in \mathcal{N}$ as f_Y is a valid pdf and hence integrable. Note that as $Var_{f_S}[s(Y)] > 0$, the distribution of $s(Y)$ is not degenerate, and hence $\mathcal{N}$ contains elements other than zero.

(ii) For $0 \leq \alpha \leq 1$, we consider $\theta = \alpha\theta_1 + (1-\alpha)\theta_2$. Then

$$\begin{aligned}\int e^{s(y)\theta} f_Y(y)\ dy &= \int \exp\{s(y)(\alpha\theta_1 + (1-\alpha)\theta_2)\} f_Y(y)\ dy \\ &= \int \exp\{s(y)\alpha\theta_1\}\exp\{s(y)(1-\alpha)\theta_2\} f_Y(y)\ dy \\ &= E_{f_Y}\left[g_1(Y;\theta_1)^\alpha g_2(Y;\theta_2)^{1-\alpha}\right]\end{aligned}$$

say, where $g_i(y;\theta) = \exp\{s(y)\theta_i\}$ for $i = 1, 2$. Now using Hölder's Inequality with $p = 1/\alpha$, $q = 1/(1-\alpha)$, we can deduce that

$$E_{f_Y}\left[g_1(Y;\theta_1)^\alpha g_2(Y;\theta_2)^{1-\alpha}\right] \leq E_{f_Y}\left[g_1(Y;\theta_1)\right]^\alpha E_{f_Y}\left[g_2(Y;\theta_2)\right]^{1-\alpha}$$

$$\int e^{s(y)\theta} f_Y(y)\ dy \leq E_{f_Y}\left[g_1(Y;\theta_1)\right]^\alpha E_{f_Y}\left[g_2(Y;\theta_2)\right]^{1-\alpha} < \infty$$

as

$$E_{f_Y}\left[g_i(Y;\theta_i)\right] = \int \exp\{s(y)\theta_i\} f_Y(y)\ dy < \infty \qquad i = 1, 2$$

Hence $\theta \in \mathcal{N}$, and the set $\mathcal{N}$ is convex.

(iii) We need to show that

$$K_S(\alpha\theta_1 + (1-\alpha)\theta_2) \leq \alpha K_S(\theta_1) + (1-\alpha)K_S(\theta_2)$$

Now, let $\theta = \alpha\theta_1 + (1-\alpha)\theta_2$. Then, using the notation from part (ii),

$$\begin{aligned} K_S(\theta) &= \log E_{f_Y}\left[g_1(Y;\theta_1)^{\alpha} g_2(Y;\theta_2)^{1-\alpha}\right] \\ &\leq \log\left\{E_{f_Y}\left[g_1(Y;\theta_1)\right]^{\alpha} E_{f_Y}\left[g_2(Y;\theta_2)\right]^{1-\alpha}\right\} \end{aligned}$$

using Hölder's Inequality again. Thus

$$\begin{aligned} K_S(\theta) &\leq \alpha \log E_{f_Y}\left[g_1(Y;\theta_1)\right] + (1-\alpha)\log E_{f_Y}\left[g_2(Y;\theta_2)\right] \\ &= \alpha K_S(\theta_1) + (1-\alpha)K_S(\theta_2) \end{aligned}$$

and the result follows.