MATH 556 - PRACTICE EXAM QUESTIONS

1. Due to the symmetry of form, this joint pdf factorizes simply as

$$f_{X,Y}(x,y) = \left\{\sqrt{c_1}\exp\left\{-\frac{x}{2}\right\}\right\}\left\{\sqrt{c_1}\exp\left\{-\frac{y}{2}\right\}\right\} = f_X(x)f_Y(y) \qquad x,y>0$$

and hence the variables are independent. Now

$$\int_0^\infty \exp\left\{-\frac{x}{2}\right\}dx = 2$$

so therefore $\sqrt{c_1} = \frac{1}{2}$, and hence $c_1 = \frac{1}{4}$.
Now random variable U, defined by

$$U = \frac{1}{2}(X-Y).$$

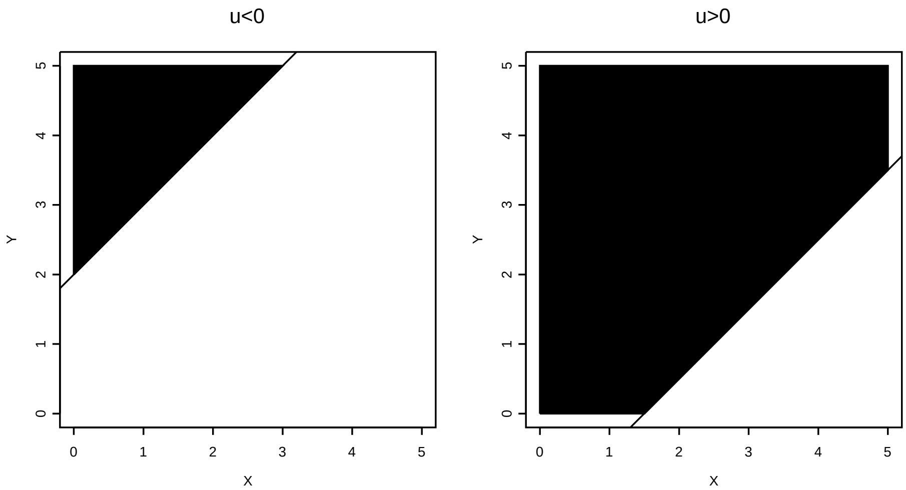
has range $\mathbb{U} \equiv R$ (X and Y are positive but unbounded random variables. Hence, for $u \in \mathbb{R}$, the cdf of U, F_U, is given by

$$F_U(u) = P[U \leq u] = P\left[\frac{1}{2}(X-Y) \leq u\right] = \iint\limits_{A_u} f_{X,Y}(x,y)\,dxdy$$

where $A_u \equiv \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : (x-y)/2 \leq u\}$. The boundary of the region A_u is determined by the three lines

$$x = 0, y = 0 \text{ and } y = x - 2u$$

This region is shaded in black in the figures below in the two cases $u < 0$ and $u \geq 0$ respectively; in these pictures the shaded region extends over all x and y above and to the left of the line $y = x - 2u$.



Integrating first dx for a fixed y, we see that the integral is always $x = 0$ to $x = y+2u$, irrespective of whether $u < 0$ or $u \geq 0$. However, the lower limit of the outer dy integral is $y = -2u$ if $u < 0$, and is zero if $u \geq 0$. Combining these together we have the lower limit of

$$l(u) = \max\{0, -2u\}$$

and hence

$$\begin{aligned}
F_U(u) &= \int_{l(u)}^{\infty} \left\{ \int_0^{y+2u} \frac{1}{4} \exp\left\{-\frac{1}{2}(x+y)\right\} dx \right\} dy \\
&= \int_{l(u)}^{\infty} \frac{1}{2} \exp\left\{-\frac{y}{2}\right\} \left\{ \int_0^{y+2u} \frac{1}{2} \exp\left\{-\frac{x}{2}\right\} dx \right\} dy \\
&= \int_{l(u)}^{\infty} \frac{1}{2} \exp\left\{-\frac{y}{2}\right\} \left[-\exp\left\{-\frac{x}{2}\right\} \right]_0^{y+2u} dy \\
&= \int_{l(u)}^{\infty} \frac{1}{2} \exp\left\{-\frac{y}{2}\right\} \left(1 - \exp\left\{-\frac{(y+2u)}{2}\right\} \right) dy \\
&= \int_{l(u)}^{\infty} \frac{1}{2} \exp\left\{-\frac{y}{2}\right\} dy - \int_{l(u)}^{\infty} \frac{1}{2} \exp\left\{-\frac{2(y+u)}{2}\right\} dy \\
&= \exp\left\{-\frac{l(u)}{2}\right\} - \frac{1}{2} \exp\{-u\} \int_{l(u)}^{\infty} \exp\{-y\}\, dy \\
&= \exp\left\{-\frac{l(u)}{2}\right\} - \frac{1}{2} \exp\{-(u + l(u))\}
\end{aligned}$$

If $u < 0$, $l(u) = -2u$, and hence

$$F_U(u) = e^u - \frac{1}{2} e^u = \frac{1}{2} e^u$$

and if $u \geq 0$, $l(u) = 0$, and hence

$$F_U(u) = 1 - \frac{1}{2} e^{-u}$$

Thus

$$f_U(u) = \begin{cases} \frac{1}{2} e^u & u < 0 \\ \frac{1}{2} e^{-u} & u \geq 0 \end{cases} = \frac{1}{2} \exp\{-|u|\} \qquad u \in \mathbb{R}$$

2. Joint pdf is constant on the ellipse $\mathcal{E}$, thus the normalizing constant is the reciprocal of the area of the ellipse, that is $1/(\pi ab)$. The range of the random variables can be re-written

$$\mathbb{X}^{(2)} \equiv \left\{(x,y) : -a < x < a, -b\left(1 - x^2/a^2\right)^{1/2} < y < b\left(1 - x^2/a^2\right)^{1/2}\right\}$$

and hence, by double integration,

$$\begin{aligned}
\iint_{\mathcal{E}} f_{X,Y}(x,y)\ dxdy &= \int_{-a}^{a}\left\{\int_{-b\left(1-x^2/a^2\right)^{1/2}}^{b\left(1-x^2/a^2\right)^{1/2}} c_2 dy\right\} dx \\
&= \int_{-a}^{a} 2c_2 b\left(1 - x^2/a^2\right)^{1/2} dx \\
&= abc_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2 t\ dt \qquad \text{(setting } x = a\sin t\text{)} \\
&= abc_2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 2t + 1)\ dt \\
&= abc_2\left[\frac{1}{2}\sin 2t + t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi abc_2
\end{aligned}$$

and hence $c_2 = 1/(\pi ab)$.

(a) For the marginal pdf of X, f_X, for fixed x,

$$f_X(x) = \int_{-b\left(1-x^2/a^2\right)^{1/2}}^{b\left(1-x^2/a^2\right)^{1/2}} \frac{1}{\pi ab} dy = \frac{2}{\pi a}\left(1 - x^2/a^2\right)^{1/2} \qquad -a < x < a,$$

(b) By symmetry of form, we must have for the marginal for Y

$$f_Y(y) = \frac{2}{\pi b}\left(1 - y^2/b^2\right)^{1/2} \qquad -b < y < b,$$

and because the two functions

$$\left(1 - x^2/a^2\right)^{1/2} \qquad\qquad \left(1 - y^2/b^2\right)^{1/2}$$

are symmetric about zero, we must have that

$$E_{f_X}[X] = E_{f_Y}[Y] = 0.$$

Finally for the covariance, we have that

$$\begin{aligned}
Cov_{f_{X,Y}}[X,Y] &= E_{f_{X,Y}}[XY] - E_{f_X}[X]\,E_{f_Y}[Y] = E_{f_{X,Y}}[XY] \\
&= \int_{-a}^{a} \left\{ \int_{-b(1-x^2/a^2)^{1/2}}^{b\left(1-x^2/a^2\right)^{1/2}} xy f_{X,Y}(x,y)\,dy \right\} dx \\
&= \int_{-a}^{a} \left\{ \int_{-b(1-x^2/a^2)^{1/2}}^{b\left(1-x^2/a^2\right)^{1/2}} y dy \right\} \frac{x}{\pi ab} dx \\
&= \int_{-a}^{a} \left[\frac{y^2}{2} \right]_{-b(1-x^2/a^2)^{1/2}}^{b\left(1-x^2/a^2\right)^{1/2}} \frac{x}{\pi ab} dx \\
&= 0
\end{aligned}$$

Hence X and Y are uncorrelated.

(c) X and Y **not** independent as there exists at least one pair $(x,y) \in \mathbb{R}^2$ such that

$$f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$$

(for example, any point within the rectangle $(-a,a) \times (-b,b)$ that is outside the ellipse has joint probability density zero, but $f_X(x) > 0$ and $f_Y(y) > 0$).

3. (a) Need expectations, variances and covariance. We have for X

$$E_{f_X}[X] = 0 \qquad E_{f_X}\left[X^2\right] = 1 \qquad Var_{f_X}[X] = 1$$

and for Y

$$E_{f_Y}[Y] = E_{f_X}\left[X^2\right] = 1 \qquad E_{f_Y}\left[Y^2\right] = E_{f_X}\left[X^4\right] = 3 \qquad Var_{f_Y}[Y] = 2$$

and for the covariance

$$E_{f_{X,Y}}[XY] = E_{f_X}\left[X^3\right] = 0 \therefore Cov_{f_{X,Y}}[X,Y] = 0 - 0 \times 1 = 0$$

and hence the correlation is also zero.

X and Y are not independent (merely uncorrelated); we have the joint distribution non-zero only on the line $y = x^2$, whereas f_X and f_Y are positive on the whole of $\mathbb{R} \times \mathbb{R}^+$.

(b) (i) By elementary properties of independent standard normal random variables (using mgfs for example)

$$X_1 - X_2 \sim Normal\,(0, 2)$$

and thus

$$Y_1 = X_1 - X_2 + 1 \sim Normal\,(1, 2)$$

(ii) By properties of the multivariate normal distribution, using multivariate transformation results

$$Y \sim N\,(b, \Sigma)$$

where

$$\Sigma = AA^T = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$$

and hence the covariance is

$$\Sigma_{12} = -2$$

and the correlation is

$$\frac{\Sigma_{12}}{\sqrt{\Sigma_{11} \times \Sigma_{22}}} = \frac{-2}{\sqrt{2 \times 4}} = -\frac{1}{\sqrt{2}}$$

4. (a) Note first that

$$f_X(x) = \frac{1}{(1+x)^2} \qquad x > 0$$

and zero otherwise. Then

$$\begin{aligned}
P[X_1 X_2 < 1] &= \int_0^\infty \int_0^{1/x_1} f_{X_1,X_2}(x_1,x_2)\, dx_2 dx_1 \\
&= \int_0^\infty \int_0^{1/x_1} \frac{1}{(1+x_1)^2} \frac{1}{(1+x_2)^2}\, dx_2 dx_1 = \int_0^\infty \left[\frac{x_2}{1+x_2}\right]_0^{1/x_1} \frac{1}{(1+x_1)^2}\, dx_1 \\
&= \int_0^\infty \frac{1/x_1}{1+1/x_1} \frac{1}{(1+x_1)^2}\, dx_1 = \int_0^\infty \frac{x_1}{(1+x_1)^3}\, dx_1 \\
&= \left[-\frac{1}{2}\frac{x_1}{(1+x_1)^2}\right]_0^\infty + \int_0^\infty \frac{1}{2}\frac{1}{(1+x_1)^2}\, dx_1 = 0 + \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

(b) Using the multivariate transformation theorem

(a) We have that $\mathbb{Y}^{(2)} \equiv \mathbb{R} \times \mathbb{R}^+$, and

$$g_1(t_1,t_2) = \frac{t_1}{\sqrt{t_1^2+t_2^2}} \qquad g_2(t_1,t_2) = \sqrt{t_1^2+t_2^2}$$

(b) Inverse transformations:

$$\left.\begin{aligned} Y_1 &= \frac{Z_1}{\sqrt{Z_1^2+Z_2^2}} \\ Y_2 &= \sqrt{Z_1^2+Z_2^2} \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} Z_1 &= Y_1 Y_2 \\ Z_2 &= \sqrt{1-Y_1^2}\,Y_2 \end{aligned}\right.$$

and thus

$$g_1^{-1}(t_1,t_2) = t_1 t_2 \qquad g_2^{-1}(t_1,t_2) = \sqrt{1-t_1^2}\,t_2$$

(c) Range: we have that $-1 < Y_1 < 1$ and $Y_2 > 0$, so $\mathbb{Y}^{(2)} = (-1,1) \times \mathbb{R}^+$

(d) The Jacobian for points $(y_1,y_2) \in \mathbb{Y}^{(2)}$ is

$$D_{y_1,y_2} = \begin{bmatrix} \dfrac{\partial z_1}{\partial y_1} & \dfrac{\partial z_1}{\partial y_2} \\ \dfrac{\partial z_2}{\partial y_1} & \dfrac{\partial z_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ \dfrac{-y_1 y_2}{\sqrt{1-y_1^2}} & \sqrt{1-y_1^2} \end{bmatrix} \Rightarrow |J(y_1,y_2)| = \frac{y_2}{\sqrt{1-y_1^2}}$$

(e) For the joint pdf we have for $(y_1,y_2) \in \mathbb{Y}^{(2)}$, by independence of Z_1 and Z_2

$$\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{Z_1,Z_2}\left(y_1 y_2, \sqrt{1-y_1^2}\,y_2\right) \times \frac{y_2}{\sqrt{1-y_1^2}} \\
&= \frac{1}{\pi}\frac{y_2 \exp\left\{-y_2^2/2\right\}}{\sqrt{1-y_1^2}}
\end{aligned}$$

and zero otherwise, where, by inspection,

$$f_{Y_1}(y_1) = \frac{1}{\pi\sqrt{1-y_1^2}} \qquad -1 < y_1 < 1 \qquad f_{Y_2}(y_2) = y_2 \exp\left\{-y_2^2/2\right\} \qquad y_2 > 0$$

Note that Y_1 and Y_2 are independent, as their joint pdf factorizes into the respective marginal pdfs at all points of $\mathbb{R}^2$.

5. (a) (i) As $n \to \infty$, for $x \in \mathbb{R}$

$$\left(\frac{1}{1+e^{-x}}\right) < 1 \qquad \therefore \qquad F_{X_n}(x) \to 0$$

and so the limiting function is not a cdf, and no limiting distribution exists.

(ii) If $U_n = X_n - \log n$. Then $\mathbb{U} \equiv (-\infty, \infty)$ and the cdf of U_n is

$$F_{U_n}(u) = P\left[U_n \le u\right] = P\left[X_n - \log n \le u\right] = P\left[X_n \le u + \log n\right] = F_{X_n}(u + \log n)$$

and so

$$F_{Y_n}(y) = \left(\frac{1}{1+e^{-u-\log n}}\right)^n = \left(\frac{1}{1+e^{-u}/n}\right)^n = \left(1 - \frac{e^{-u}}{n+e^{-u}}\right)^n$$

Thus as $n \to \infty$, for all u

$$F_{U_n}(u) \to \exp\left\{-e^{-u}\right\} \qquad \therefore \qquad F_{U_n}(u) \to F_U(u) = \exp\left\{-e^{-u}\right\}$$

and the limiting distribution of U_n does exist, and is continuous on $\mathbb{R}$.

Thus, for large n,

$$P[X_n > k] = P[U_n > k + \log n] = 1 - F_{U_n}(k + \log n) \approx 1 - F_U(k + \log n) = 1 - \exp\left\{-e^{-k-\log n}\right\}$$

(b) Let X_i denote the score on roll i. Then

$$E_{f_{X_i}}[X_i] = \frac{-2 + (4 \times -1) + 6}{6} = 0 \qquad Var_{f_{X_i}}[X_i] = E_{f_{X_i}}[X_i^2] = \frac{4 + (4 \times 1) + 36}{6} = \frac{22}{3}$$

and denote these quantities μ and σ^2 respectively.

(i) The expectation and variance of T_{100} are $100\mu = 0$ and $100\sigma^2 = 2200/3$.

(ii) The Central Limit Theorem gives that for the iid $\{X_i\}$ collection

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} \sim AN(0,1)$$

where AN denotes Asymptotically Normal (as $n \to \infty$). Thus

$$T_n = \sum_{i=1}^{n} X_i \sim AN\left(0, \frac{22n}{3}\right)$$

and

(iii) Using the Weak Law of Large numbers, we can deduce that

$$M_n \xrightarrow{p} \mu = 0$$

as $n \longrightarrow \infty$, that is, the sample mean random quantity converges in probability to zero, that is, the probability distribution of M_n becomes degenerate at zero.

6. (a) (i) The Poisson distribution mgf is

$$M_X(t) = \exp\left\{\lambda(e^t - 1)\right\}.$$

Now, if $Z_\lambda = (X - \lambda)/\sqrt{\lambda}$, we use the mgf result for linear functions, that is if

$$Y = aX + b \Longrightarrow M_Y(t) = e^{bt} M_X(at).$$

Here, $a = 1/\sqrt{\lambda}$ and $b = -\sqrt{\lambda}$, so

$$\begin{aligned} M_{Z_\lambda}(t) &= e^{-\sqrt{\lambda}t} \exp\left\{\lambda(e^{t/\sqrt{\lambda}} - 1)\right\} = \exp\left\{-\lambda^{1/2}t + \lambda\left[\frac{t}{\lambda^{1/2}} + \frac{t^2}{2\lambda} + \frac{t^3}{6\lambda^{3/2}} + \ldots\right]\right\} \\ &= \exp\left\{\frac{t^2}{2} + \frac{t^3}{6\sqrt{\lambda}} + \ldots\right\} \to \exp\left\{\frac{t^2}{2}\right\} \qquad \text{as } \lambda \to \infty \end{aligned}$$

so therefore

$$Z_\lambda \xrightarrow{d} Z \sim Normal(0, 1)$$

as $\lambda \to \infty$.

(ii) Let $T_i = X_i + Y_i$. Then, by properties of Poisson random variables, we have that $T_i \sim Poisson\left(\lambda_X + \lambda_Y\right)$. Hence

$$T = \sum_{i=1}^{n} (X_i + Y_i) \sim Poisson\left(n\left(\lambda_X + \lambda_Y\right)\right).$$

so that

$$E_{f_T}[T] = n\left(\lambda_X + \lambda_Y\right) \qquad Var_{f_T}[T] = n\left(\lambda_X + \lambda_Y\right)$$

But $M = \dfrac{T}{n}$, so

$$E_{f_M}[M] = \frac{n\left(\lambda_X + \lambda_Y\right)}{n} = \lambda_X + \lambda_Y \qquad Var_{f_M}[M] = \frac{n\left(\lambda_X + \lambda_Y\right)}{n^2}$$

which are both finite. Hence, by the *Weak Law of Large Numbers*

$$M \xrightarrow{p} E_{f_M}[M] = \lambda_X + \lambda_Y = \mu$$

(b) (i) $T_n = \max\{X_1, \ldots, X_n\}$ so

$$F_{T_n}(t) = \{F_X(t)\}^n = \left(1 - e^{-\lambda t}\right)^n \qquad t \in \mathbb{R}^+$$

(ii) In the limit as $n \to \infty$ we have the limit for *fixed* t as

$$F_{T_n}(t) \to 0 \qquad \text{for all } t$$

Hence there is *no limiting distribution*.

(iii) If $U_n = \lambda T_n - \log n$, we have from first principles that for $u > -\log n$

$$\begin{aligned} F_{U_n}(u) &= P\left[U_n \leq u\right] = P\left[\lambda T_n - \log n \leq u\right] \\ &= P\left[T_n \leq \frac{1}{\lambda}\left(u + \log n\right)\right] \\ &= F_{T_n}\left(\frac{1}{\lambda}\left(u + \log n\right)\right) \\ &= \left(1 - e^{-(u+\log n)}\right)^n \\ &= \left(1 - \frac{e^{-u}}{n}\right)^n \end{aligned}$$

so that

$$F_{U_n}(u) \to \exp\left\{-e^{-u}\right\} \qquad \text{as } n \to \infty$$

which is a valid cdf. Hence the limiting distribution is

$$F_U(u) = \exp\left\{-e^{-u}\right\} \qquad u \in \mathbb{R}$$