MATH 556 - EXERCISES 4

These exercises are not for assessment

1. Using the Central Limit Theorem, construct Normal approximations to probability distribution of a random variable X having

   (i) a Binomial distribution, $X \sim Binomial(n, \theta)$
   (ii) a Poisson distribution, $X \sim Poisson(\lambda)$
   (iii) a Negative Binomial distribution, $X \sim NegBinomial(n, \theta)$
   (iv) a Gamma distribution,$X \sim Gamma(\alpha, \beta)$

In the following questions, use the following results concerning extreme *order statistics*; let Y_n and Z_n correspond to the *maximum* and *minimum* order statistics derived from random sample $X_1, ...X_n$ from population with cdf F_X, that is

$$Y_n = \max\{X_1, ..., X_n\} \qquad Z_n = \min\{X_1, ..., X_n\}.$$

Then the cdfs of Y_n and Z_n are given by

$$F_{Y_n}(y) = \{F_X(y)\}^n \qquad F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n.$$

2. Suppose $X_1, ..., X_n \sim Uniform(0, 1)$, that is

$$F_X(x) = x \qquad 0 \le x \le 1$$

   Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$.

3. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = 1 - x^{-1} \qquad x \ge 1$$

   Find the cdfs of Z_n and $U_n = Z_n^n$, and the limiting distributions of Z_n and U_n as $n \longrightarrow \infty$.

4. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = \frac{1}{1 + e^{-x}} \qquad x \in \mathbb{R}$$

   Find the cdfs of Y_n and $U_n = Y_n - \log n$ and the limiting distributions of Y_n and U_n as $n \longrightarrow \infty$.

5. Suppose $X_1, ..., X_n$ have cdf

$$F_X(x) = 1 - \frac{1}{1 + \lambda x} \qquad x > 0$$

   Find the cdfs of Y_n and Z_n, and the limiting distributions as $n \longrightarrow \infty$. Find also the cdfs of $U_n = Y_n/n$ and $V_n = nZ_n$, and the limiting distributions of U_n and V_n as $n \longrightarrow \infty$.

6. Suppose $X_1, ..., X_n \sim Poisson(\lambda)$ are independent random variables. Let

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

   Show that $M_n \xrightarrow{p} \lambda$ as $n \longrightarrow \infty$. If random variable T_n is defined by $T_n = e^{-M_n}$, show that $T_n \xrightarrow{p} e^{-\lambda}$, and find an approximation to the probability distribution of T_n as $n \longrightarrow \infty$.

7. For the following sequences of random variables, $\{X_n\}$, decide whether the the sequence converges in *mean-square* (rth mean for $r = 2$) or *in probability* as $n \longrightarrow \infty$.

(a) $$X_n = \begin{cases} 1 & \text{with prob. } 1/n \\ 2 & \text{with prob. } 1 - 1/n \end{cases}$$

(b) $$X_n = \begin{cases} n^2 & \text{with prob. } 1/n \\ 1 & \text{with prob. } 1 - 1/n \end{cases}$$

(c) $$X_n = \begin{cases} n & \text{with prob. } 1/\log n \\ 0 & \text{with prob. } 1 - 1/\log n \end{cases}$$

Almost sure convergence and the Borel-Cantelli Lemma.

8. Consider the sequence of random variables defined for $n = 1, 2, 3, ...$ by

$$X_n = I_{[0,n^{-1})}(U_n)$$

where $U_1, U_2, ...$ are a sequence of independent $Uniform(0,1)$ random variables, and I_A is the indicator function for set A

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Does the sequence $\{X_n\}$ converge

(a) almost surely ?
(b) in r^{th} mean for $r = 1$?

[*Hint: Consider the events* $A_n \equiv (X_n \neq 0)$ *for* $n = 1, 2, \ldots$*]*

9. Let $Z \sim Uniform(0,1)$, and define a sequence of random variables $\{X_n\}$ by

$$X_n = nI_{[1-n^{-1},1)}(Z) \qquad n = 1, 2, \ldots$$

where, for set A

$$I_A(Z) = \begin{cases} 1 & Z \in A \\ 0 & Z \notin A \end{cases}$$

that is, I_A is the indicator random variable associated with the set A.

Does the sequence $\{X_n\}$ converge in any mode to any limit random variable ? Justify your answer.

10. Suppose, for $n = 1, 2, \ldots$, $X_n \sim Bernoulli(p_n)$ are a sequence of independent random variables where

$$P[X_n = 1] = p_n = \frac{1}{\sqrt{n}}.$$

Does $P[\ X_n = 1 \text{ infinitely often }] = 1$? Justify your answer.