MATH 556 - EXERCISES 3

These exercises are not for assessment

1. The joint pdf $f_{X,Y}$ of positive random variables X and Y is specified as

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$$

Identify the marginal distribution of Y when

(i) $Y|X = x \sim Poisson(x)$ and $X \sim Gamma(\alpha, \beta)$.
(ii) $Y|X = x \sim Exponential(x)$ and $X \sim Gamma(\alpha, \beta)$.
(iii) $Y|X = x \sim Binomial(n, x)$ and $X \sim Beta(\alpha, \beta)$.

2. Suppose that Y has a finite mixture distribution where

$$f_Y(y) = \sum_{k=1}^{K} \pi_k f_k(y|\theta_k)$$

for component pmfs/pdfs $f_1, \ldots, f_k$ and $(\pi_1, \ldots, \pi_K)$ satisfy $0 < \pi_k < 1$ and $\sum_{k=1}^{K} \pi_k = 1$. Let $\mu_1, \ldots, \mu_K$ be the expected values derived from each of the component pmfs/pdfs, and let $M_1, \ldots, M_K$ be the corresponding mgfs. Show that

$$E_{f_Y}[Y] = \sum_{k=1}^{K} \pi_k \mu_k$$

and find a similar representation for the mgf $M_Y(t)$.

3. A simple and valid finite mixture distribution with $K = 2$ can be constructed by setting

$$f_1(y) = f_{Y|X}(y|x = 1) = I_{\{0\}}(y) \qquad f_2(y) = f_{Y|X}(y|x = 2) \equiv Normal(0, 1)$$

where $I_A(x)$ is the indicator function for set A. Suppose that

$$f_X(1) = P[X = 1] = \pi \qquad f_X(2) = P[X = 2] = 1 - \pi$$

Suppose that it is **known** that $y = 0$. Using Bayes Theorem, derive the *posterior probability* conditional on $y = 0$, that is

$$f_{X|Y}(x|0)$$

for $x = 1, 2$.

4. The *skewness*, ς, and *kurtosis*, κ, of a probability distribution f_X are defined by

$$\varsigma = \frac{E_{f_X}[(X - \mu)^3]}{\sigma^3} \qquad \kappa = \frac{E_{f_X}[(X - \mu)^4]}{\sigma^4}$$

where μ and σ^2 are the expectation and variance of f_X.

(i) Compute the skewness and kurtosis for the $Normal(0, 1)$ distribution.
(ii) *Scale Mixtures:* Suppose that $X|V = v \sim N(0, v)$ and $V \sim f_V$, where V is a positive random variable. Using iterated expectation, give an expression for the form of the (marginal) skewness and kurtosis of X.
(iii) *Location-Scale Mixtures:* Show that a distribution with skewness not equal to zero can be constructed using a location-scale mixture of a $Normal(0, 1)$ pdf.

5. Find the *skewness* of the following standard distributions

(i) $Bernoulli(\theta)$
(ii) $Poisson(\lambda)$
(iii) $Gamma(\alpha, \beta)$