MATH 556 - ASSIGNMENT 4 SOLUTIONS

1 For $t > 0$, and constant $k > 0$

$$P[X \geq t] = P[X + k \geq t + k] \leq P[(X+k)^2 \geq (t+k)^2] \leq \frac{E_{f_X}[(X+k)^2]}{(t+k)^2}$$

by the Chebychev Lemma. Now if $k = \sigma^2/t$, then

$$\begin{aligned} P[X \geq t] \leq \frac{E_{f_X}[(X+\sigma^2/t)^2]}{(t+\sigma^2/t)^2} = \frac{E_{f_X}[(tX+\sigma^2)^2]}{(t^2+\sigma^2)^2} &= \frac{t^2 E_{f_X}[X^2] + 2tE_{f_X}[X] + \sigma^4}{(t^2+\sigma^2)^2} \\ &= \frac{t^2\sigma^2 + \sigma^4}{(t^2+\sigma^2)^2} \end{aligned}$$

as $E_{f_X}[X] = 0$ implies $\sigma^2 = E_{f_X}[X^2]$. Thus

$$P[X \geq t] \leq \frac{\sigma^2}{t^2+\sigma^2}$$

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2 If $Y_n = \max\{X_1, \ldots, X_n\}$, then for $y \in \mathbb{R}$,

(a) For Y_n

$$F_{Y_n}(y) = \{F_X(y)\}^n = \left(\frac{1}{2} + \frac{1}{\pi}\arctan(x)\right)^n.$$

But clearly, for any y, $F_{Y_n}(y) \longrightarrow 0$, so there is no limiting distribution.

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(b) If $T_n = \pi Y_n/n$, then

$$F_{T_n}(y) = \{F_X(ny/\pi)\}^n = \left(\frac{1}{2} + \frac{1}{\pi}\arctan(ny/\pi)\right)^n$$

- For $y < 0$,

$$\arctan(ny/\pi) < 0 \qquad \therefore \qquad \left(\frac{1}{2} + \frac{1}{\pi}\arctan(ny/\pi)\right) < 1/2$$

  and thus for $y < 0$, $F_{T_n}(y) \longrightarrow 0$ as $n \longrightarrow \infty$.

- For $y = 0$, $\arctan(0) = 0$, so

$$F_{T_n}(0) = \left(\frac{1}{2}\right)^n \longrightarrow 0$$

  and so $F_{T_n}(y) \longrightarrow 0$ as $n \longrightarrow \infty$.

- For $y > 0$,

$$\frac{1}{2} + \frac{1}{\pi}\arctan(ny/\pi) = 1 + \frac{1}{\pi}\arctan(-\pi/(ny)) = 1 + \frac{1}{\pi}\left[-\frac{\pi}{ny} + o(n^{-1})\right]$$

  using the approximation given. Thus

Hence, for $y > 0$, as $n \longrightarrow \infty$,

$$F_{T_n}(y) = \left(1 - \frac{1}{\pi}\left[\frac{\pi}{ny} + o(n^{-1})\right]\right)^n = \left(1 - \frac{1}{ny} + o(n^{-1})\right)^n \longrightarrow \exp\{-1/y\}$$

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Note that for arbitrary A, B,

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

so with $A = \pi/2, \cos A = 0, \sin A = 1$, so

$$\tan(\pi/2 + B) = -\frac{\cos B}{\sin B} = -\frac{1}{\tan B}.$$

Thus, if $\tan B = x$, then

$$\tan(\pi/2 + \arctan(x)) = -\frac{1}{x} \qquad \therefore \qquad \tan(\pi/2 + \arctan(-1/x)) = x$$

and thus

$$\pi/2 + \arctan(-1/x) = \arctan(x).$$

Hence, with $x = ny/\pi$,

$$\frac{1}{2} + \frac{1}{\pi}\arctan(ny/\pi) = 1 + \frac{1}{\pi}\arctan(-\pi/(ny))$$

3 For $x \in \mathbb{R}$

$$f_{X_n}(x) = \frac{1}{\pi}\frac{n}{1+n^2x^2} \qquad x \in \mathbb{R}.$$

(i) Convergence in rth mean to zero;

$$E_{f_{X_n}}[|X_n|^r] = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{n|x|^r}{1+n^2x^2} = \frac{2}{\pi}\int_0^{\infty} \frac{nx^r}{1+n^2x^2}$$

This integral is **divergent** if $r \geq 1$, but **convergent** to zero as $n \longrightarrow \infty$ if $0 < r < 1$. Thus, if we only consider r to take integer values, there is no convergence, but for $0 < r < 1$, $X_n \xrightarrow{r} X$.

(ii) Convergence in probability; for $\epsilon > 0$,

$$P[|X_n| < \epsilon] = F_{X_n}(\epsilon) - F_{X_n}(-\epsilon) = \frac{1}{\pi}\arctan(n\epsilon) - \frac{1}{\pi}\arctan(-n\epsilon) \longrightarrow 1$$

as $n \longrightarrow \infty$. Hence $X_n \xrightarrow{p} X$.

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Of course

$$X_n \xrightarrow{r} X \text{ , some } r > 0 \qquad \Longrightarrow \qquad X_n \xrightarrow{p} X$$

by general relationships between the modes of convergence.

4 Suppose $X_n \stackrel{p}{\longrightarrow} 0$, so that for any $\epsilon > 0$,

$$\lim_{n \longrightarrow \infty} P[|X_n| > \epsilon] = 0$$

Now by iterated expectation, conditioning in turn on the partitioning events

$$(|X_n| \leq \epsilon) \qquad (|X_n| > \epsilon)$$

we have

$$\begin{aligned} E\left[\frac{|X_n|}{1+|X_n|}\right] &= E\left[\frac{|X_n|}{1+|X_n|} \mid |X_n| \leq \epsilon\right] P[|X_n| \leq \epsilon] + E\left[\frac{|X_n|}{1+|X_n|} \mid |X_n| > \epsilon\right] P[|X_n| > \epsilon] \\ &\leq \frac{\epsilon}{1+\epsilon} \times P[|X_n| \leq \epsilon] + 1 \times P[|X_n| > \epsilon] \\ &\longrightarrow \frac{\epsilon}{1+\epsilon} \qquad \text{as } n \longrightarrow \infty. \end{aligned}$$

But this holds for arbitrary $\epsilon > 0$, so

$$E\left[\frac{|X_n|}{1+|X_n|}\right] \longrightarrow 0.$$

Conversely, suppose

$$E\left[\frac{|X_n|}{1+|X_n|}\right] \longrightarrow 0.$$

Then, using the Chebychev Lemma and the hint

$$P[|X_n| > \epsilon] = P\left[\frac{|X_n|}{1+|X_n|} > \frac{\epsilon}{1+\epsilon}\right] \leq \left(\frac{1+\epsilon}{\epsilon}\right) E\left[\frac{|X_n|}{1+|X_n|}\right] \longrightarrow 0$$

as $n \longrightarrow \infty$. Thus

$$P[|X_n| > \epsilon] \longrightarrow 0 \qquad \therefore \qquad X_n \stackrel{p}{\longrightarrow} 0$$

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