MATH 556 - ASSIGNMENT 3 SOLUTIONS

1 (a) By iterated expectation, using the formula sheet to quote expectations for Gamma and Poisson

$$E_{f_X}[X] = E_{f_N}[E_{f_{X|N}}[X|N=n]] = E_{f_N}\left[\frac{N+r/2}{1/2}\right] = \frac{E_{f_N}[N]+r/2}{1/2} = \frac{\lambda + r/2}{1/2} = 2\lambda + r$$

3 MARKS

(b) By the same method of iterated expectation, for $-1/2 < t < 1/2$,

$$\begin{aligned}
M_X(t) = E_{f_X}[e^{tX}] &= E_{f_N}[E_{f_{X|N}}[e^{tX}|N=n]] = E_{f_N}\left[\left(\frac{1/2}{1/2-t}\right)^{N+r/2}\right] \\
&= \left(\frac{1/2}{1/2-t}\right)^{r/2} E_{f_N}\left[\left(\frac{1/2}{1/2-t}\right)^{N}\right] \\
&= \left(\frac{1}{1-2t}\right)^{r/2} G_N\left(\frac{1}{1-2t}\right) \\
&= \left(\frac{1}{1-2t}\right)^{r/2} \exp\left\{\lambda\left(\frac{1}{1-2t}-1\right)\right\} \\
&= \left(\frac{1}{1-2t}\right)^{r/2} \exp\left\{\frac{2\lambda t}{1-2t}\right\}
\end{aligned}$$

6 MARKS

2 (a) By direct calculation the mgf of $Y_i = X_i^2$ is

$$\begin{aligned}
M_{Y_i}(t) &= E_{f_{X_i}}[e^{tX_i^2}] = \int_{-\infty}^{\infty} e^{tx^2}\left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}(x-\mu_i)^2\right\} dx \\
&= \left(\frac{1}{1-2t}\right)^{1/2} \exp\left\{\frac{\mu_i^2 t}{1-2t}\right\}
\end{aligned}$$

whenever $-1/2 < t < 1/2$, after completing the square in x in the exponent and integrating the result, in which the integrand is proportional to a normal pdf. Hence, using the result for independent rvs,

$$M_Y(t) = \prod_{i=1}^{r} M_{Y_i}(t) = \left(\frac{1}{1-2t}\right)^{r/2} \exp\left\{\frac{\theta t}{1-2t}\right\}$$

where $\theta = \sum_{i=1}^{r} \mu_i^2$.

6 MARKS

*Note: this is the same form as for Q1, (b), with $\lambda = \theta/2$. The distribution of Y here is the **non-central Chisquared distribution** with r degrees of freedom and non-centrality parameter μ.*

(b) Many possible routes to compute the result. Could differentiate the mgf, or use direct calculation, or differentiate the cumulant generating function three times and evaluate at zero;

$$K_Y(t) = \log M_Y(t) = -\frac{r}{2}\log(1-2t) + \frac{\theta t}{1-2t}$$

so

$$K_Y^{(1)}(t) = \frac{r}{1-2t} + \frac{(1-2t)\theta + 2\theta t}{(1-2t)^2} = \frac{r}{1-2t} + \frac{\theta}{(1-2t)^2}$$

so that $\mu = E_{f_Y}[Y] = K_Y^{(1)}(0) = r + \theta$.

$$K_Y^{(2)}(t) = \frac{2r}{(1-2t)^2} + \frac{4\theta}{(1-2t)^3}$$

so that $\sigma^2 = Var_{f_Y}[Y] = K_Y^{(2)}(0) = 2r + 4\theta = 2(r + 2\theta)$. Finally,

$$K_Y^{(3)}(t) = \frac{8r}{(1-2t)^3} + \frac{24\theta}{(1-2t)^4}$$

so that

$$E_{f_Y}[(Y-\mu)^3] = K_Y^{(3)}(0) = 8r + 24\theta$$

yielding that

$$\varsigma = \frac{E_{f_Y}[(Y-\mu)^3]}{\sigma^3} = \frac{8r + 24\theta}{(2r+4\theta)^{3/2}} = \frac{2^{3/2}(r+3\theta)}{(r+2\theta)^{3/2}}$$

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It is easy to verify that $K_X^{(3)}(0) = E_{f_X}[(X-\mu)^3]$ by direct evaluation, complementing the earlier results that $K_X^{(1)}(0) = E_{f_X}[X]$ and $K_X^{(2)}(0) = E_{f_X}[(X-\mu)^2]$.

3 As

$$S_{i+1} = \sum_{j=1}^{s_i} N_{ij} + K_i$$

with all variables independent, we have immediately using the result from lectures, and properties of pgfs, that

$$G_{i+1}(t) = G_i(G_N(t))G_K(t) = G_N(G_i(t))G_K(t)$$

where G_i is the pgf of S_i.

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Note that

$$G_N(G_i(t)) = G_N(G_N(G_{i-1}(t))) = \cdots = G_N(G_N(\ldots G_N(t)\ldots))$$

*iterating i times **inside**, but taking the $i-1$ **outer** computations together yields*

$$G_{i-1}(G_N(t))$$