MATH 556 - ASSIGNMENT 2 SOLUTIONS

1 (a) (i) By direct calculation

$$C_X(t) = E_{f_X}[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} \exp\{-x - e^{-x}\}\, dx \;=\; \int_{-\infty}^{\infty} \exp\{-(1-it)x - e^{-x}\}\, dx$$
$$= \int_0^{\infty} y^{-it} e^{-y}\, dx = \Gamma(1-it)$$

after setting $y = e^{-x}$.

Note that the Gamma function notation usage here is legitimate; the Gamma function is defined for complex arguments. However, the final step of computing the integral is technically more complicated than it appears, as it involves complex terms; fortunately the mgf exists in a neighbourhood of zero, so we can mimic the entire computation by looking at the mgf, M_X, and then substituting in it for t at the last line.

4 MARKS

(ii) By direct calculation, using the series expansion and integrating term by term;

$$\begin{aligned} C_X(t) = E_{f_X}[e^{itX}] &= \int_{-\infty}^{\infty} e^{itx} \sum_{k=0}^{\infty} (-1)^k \exp\{-(2k+1)\pi|x|\}\, dx \\ &= \sum_{k=0}^{\infty} (-1)^k \int_{-\infty}^{\infty} e^{itx} \exp\{-(2k+1)\pi|x|\}\, dx \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi} \int_{-\infty}^{\infty} e^{i(t/(2k+1)\pi)y} \exp\{-|y|\}\, dy \end{aligned}$$

setting $y = (2k+1)\pi x$, after exchanging the order of integration and differentiation, which is legitimate in this context as the sum and integral are convergent. Using the result from lectures computing the cf for the Double Exponential distribution,

$$\int_{-\infty}^{\infty} e^{ity} \frac{1}{2} \exp\{-|y|\}\, dy = \frac{1}{1+t^2} \qquad \therefore \qquad \int_{-\infty}^{\infty} e^{i\frac{t}{(2k+1)\pi}y} \exp\{-|y|\}\, dy = \frac{2}{1 + \left\{ \dfrac{t}{(2k+1)\pi} \right\}^2}$$

so that

$$C_X(t) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi} \frac{2}{1 + \left\{ \dfrac{t}{(2k+1)\pi} \right\}^2} = \sum_{k=0}^{\infty} (-1)^k \frac{2(2k+1)\pi}{(2k+1)^2\pi^2 + t^2} \qquad (1)$$

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Proof to here is sufficient for full marks. Again, the mgf exists, so we can mimic the calculation of C_X by actually computing M_X and substituting.

In fact, C_X can be obtained in closed form; it can be shown using advanced methods (possibly complex analysis; some more details will appear on the course website) that

$$C_X(t) = \frac{2e^{t/2}}{e^{t/2} + e^{-t/2}} = \frac{1}{\cosh(t/2)} \qquad t \in \mathbb{R}$$

and this result can be verified using the inversion formula. It is then clear that the pdf and its cf are identical in form, and straightforward to verify that this distribution is infinitely divisible.

There is a comprehensive list of cfs for probability distributions in the famous book by Feller (Vol II). Two other very useful books for looking up awkward integrals are also listed below.

- W. Feller *An Introduction to Probability Theory and Its Applications, Volume 2*, (1971), John Wiley.
- M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables* (1965), Dover Publications.
- I. S. Gradshteyn, I. M. Ryzhik et al., *Table of Integrals, Series, and Products, Sixth Edition* (2000), Academic Press.

For this part of the assignment, I should have made clear that proof to the form in equation (1) - after spotting the connection to the Double Exponential - was sufficient. I apologize for omitting this, and hope that you did not waste too much time trying to evaluate the sum.

(iii) The cf is integrable, so the inversion formula for continuous pdfs can be used. Hence

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} C_X(t)\ dt \quad = \quad \frac{1}{2\pi}\int_{-1}^{1} e^{-itx}(1-|t|)\ dt = \frac{1}{\pi}\int_{0}^{1}(1-t)\cos(tx)\ dt$$

after writing $e^{-itx} = \cos(tx) - i\sin(tx)$, and splitting the integral into two halves over $(-1 < t < 0)$ and $(0 < t < 1)$ in the usual way. Integrating by parts yields

$$f_X(x) = \frac{1}{\pi x^2}(1-\cos x) \qquad x \in \mathbb{R}$$

It is straightforward to verify that this is a valid pdf.

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(b) Could use the general inversion formula to deduce the form of the distribution. However, C_X is in the form of a pure $\cos$ function, and therefore can be written as the sum of complex exponentials, that is

$$C_X(t) = \cos(\theta t) = \frac{1}{2}\left[(\cos(\theta t) + i\sin(\theta t)) + (\cos(\theta t) - i\sin(\theta t))\right] = \frac{1}{2}e^{it\theta} + \frac{1}{2}e^{-it\theta}$$

and hence it follows that X has a discrete distribution with pmf given by

$$f_X(x) = \begin{cases} \dfrac{1}{2} & x = -\theta \\[2ex] \dfrac{1}{2} & x = \theta \\[2ex] 0 & \text{otherwise} \end{cases}$$

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2 If $X \sim Cauchy$, then $Y = 1/X$ has pdf given by the general transformation theorem

$$f_Y(y) = f_X(1/y) \times |J(y)| = \frac{1}{\pi}\frac{1}{1+(1/y)^2} \ \times \ |-1/y^2| = \frac{1}{\pi}\frac{1}{1+y^2} \qquad y \in \mathbb{R}$$

so in fact $Y \sim Cauchy$ also. Now, the sample mean rv $\overline{X}$ has characteristic function given by

$$\{C_X(t/n)\}^n = \{\exp\{-|t/n|\}\}^n = \exp\{-|t|\}$$

so $\overline{X}$ also has a $Cauchy$ distribution. Combining these results gives that

$$Z_n = \frac{1}{\overline{X}} \sim Cauchy.$$

The cdf of the Cauchy distribution takes the form

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi} \frac{1}{1+y^2} dy = \frac{1}{\pi} \arctan(x) + \frac{1}{2} \qquad x \in \mathbb{R}$$

and hence

$$P[|Z_n| \leq c] = F_X(c) - F_X(-c) = \frac{1}{\pi} \arctan(c) - \frac{1}{\pi} \arctan(-c) = \frac{2}{\pi} \arctan(c)$$

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3 From the course formula sheet, if $X \sim Gamma(\alpha, \beta)$, then the mgf for X is given by

$$M_X(t) = \left(\frac{\beta}{\beta - t} \right)^{\alpha} \qquad -\beta < t < \beta,$$

say. If we take

$$Z_{nj} \sim Gamma(\alpha/n, \beta) \qquad j = 1, \ldots, n$$

as a collection of iid variables, and define Z_n as their sum, then the mgf of Z_n is

$$\left\{ \left(\frac{\beta}{\beta - t} \right)^{\alpha/n} \right\}^n = \left(\frac{\beta}{\beta - t} \right)^{\alpha}$$

and Z_n and X have the same distribution. We compute the corresponding cfs by substitution. Hence X has an infinitely divisible distribution.

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