Testing in the RBD with Replication

The three F statistics

$$F = \frac{MST}{MSE} \quad F = \frac{MSB}{MSE} \quad F = \frac{MSI}{MSE}$$

can be used to test for significant Treatment, Block and Interaction effects respectively.

Now

$$MSE = \frac{SSE}{\text{Error d.f.}}$$

But what is "Error d.f." ? It is a constant that dictates how large SSE should be on average.

The general rule for computing the error d.f. for any model is

$$\text{Error d.f.} = n - p$$

where n is the total **sample size** and p is the total **number of parameters fitted**.

How many parameters do we fit ?

- No Interaction

$$p = 1 + (b - 1) + (k - 1)$$

  that is, the overall mean μ, plus the $b - 1$ differences from μ due to the blocks, plus the $k - 1$ differences from μ due to the treatments.

- Interaction

$$p = bk$$

  that is, one parameter in each cell of the two-way table of blocks by treatments.

Thus

- No Interaction

  $$p = 1 + (b - 1) + (k - 1) = b + k - 1$$

  parameters, so

  $$\text{Error d.f.} = n - p = n - b - k + 1$$

- Interaction: we fit $p = bk$ parameters, so

  $$\text{Error d.f.} = n - p = n - bk$$

It transpires that if

$$MSI = \frac{SSI}{(b-1)(k-1)}$$

is the Mean Square for Interaction, then

$$F = \frac{MSI}{MSE}$$

yields a test statistic suitable for testing interaction. If there is **no interaction**, then

$$F \sim \text{Fisher-F}((b-1)(k-1), n-bk)$$

where $n = bkr$.

Why $(b-1)(k-1)$? This is the number of **extra** parameters we fit to include the interaction.

For the CRD:

$$\begin{array}{rcl} H_a & & H_0 \\ \text{FULL MODEL} & \longrightarrow & \text{NULL MODEL} \\ k \text{ parameters} & \longrightarrow & 1 \text{ parameter} \end{array}$$

so there are $(k-1)$ extra parameters, and SST varies on $(k-1)$ degrees of freedom.

For the RBD: the (i, j)th treatment/block combination has mean

$$\mu_i + \mu_j^B$$

so for testing for a TREATMENT effect

$$\begin{array}{rcl} H_a & & H_0 \\ \text{FULL MODEL} & \longrightarrow & \text{NULL MODEL} \\ k \text{ parameters} & \longrightarrow & 1 \text{ parameter} \end{array}$$

so there are $(k-1)$ extra parameters, and SST varies on $(k-1)$ degrees of freedom.

$$\mu_1, \ldots, \mu_k \longrightarrow \mu$$

For testing for a BLOCK effect

$$\begin{array}{rcl} H_a & & H_0 \\ \text{FULL MODEL} & \longrightarrow & \text{NULL MODEL} \\ b \text{ parameters} & \longrightarrow & 1 \text{ parameter} \end{array}$$

so there are $(b-1)$ extra parameters, and SSB varies on $(b-1)$ degrees of freedom.

$$\mu_1^{(B)}, \ldots, \mu_k^{(B)} \longrightarrow \mu^{(B)}$$

These models and tests can be fitted and carried out even if we do not have replication.

With replication, we can investigate the interaction, that is the model where the (i, j)th treatment/block combination has mean

$$\mu_i + \mu_j^B + \mu_{ij}$$

rather than the model where

$$\mu_i + \mu_j^B$$

that is, we wish to test

$$\begin{aligned} H_0 &: \mu_{ij} = 0 \qquad \text{for all } i \text{ and } j \\ H_a &: \mu_{ij} \neq 0 \end{aligned}$$

In the **full interaction** model: we fit bk parameters

In the **restricted**, **no interaction** model: we fit

$$1 + (b - 1) + (k - 1) = b + k - 1$$

parameters. Therefore the differences is

$$bk - (b + k - 1) = bk - b - k + 1 = (b - 1)(k - 1)$$

and SSI varies on $(b - 1)(k - 1)$ degrees of freedom.

ANOVA Table

SOURCE	DF	SS	MS	F
TMTS	$k-1$	SST	MST	F_T
BLOCKS	$b-1$	SSB	MSB	F_B
INTERACTION	$(b-1)(k-1)$	SSI	MSI	F_I
ERROR	$(n-bk)$	SSE	MSE	
TOTAL	$n-1$	SS		

where

$$MST = \frac{SST}{k-1} \qquad MSB = \frac{SSB}{b-1}$$

$$MSI = \frac{SST}{(b-1)(k-1)} \qquad MSE = \frac{SSE}{n-bk}$$

and

$$F_T = \frac{MST}{MSE} \qquad F_B = \frac{MSB}{MSE} \qquad F_I = \frac{MSI}{MSE}$$

Example: Batteries Data (see handout)

Tests of Between-Subjects Effects

Dependent Variable: Battery Life (hr)

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	59154.000[a]	8	7394.250	11.103	.000
Intercept	398792.250	1	398792.250	598.829	.000
temp	39083.167	2	19541.583	29.344	.000
material	10633.167	2	5316.583	7.983	.002
temp * material	9437.667	4	2359.417	3.543	.019
Error	17980.750	27	665.954		
Total	475927.000	36			
Corrected Total	77134.750	35			

a. R Squared = .767 (Adjusted R Squared = .698)

For $\alpha = 0.05$, there is a significant **temp** effect ($p < 0.001$), and a significant **material** effect ($p = 0.002$), and a significant interaction ($p = 0.019$)

NB: If we do not have replication, we CANNOT fit the interaction. Recall that

$$\text{Error d.f.} = n - bk$$

but if $r = 1$, $n = rbk = bk$, so the error d.f. is zero.

In fact, $SSE = 0$ also, so the MSE is not defined.