MATH 556: MATHEMATICAL STATISTICS I
SOME EXAMPLES OF CHARACTERISTIC FUNCTIONS

The characteristic function for a random variable X with pmf/pdf f_X is defined for $t \in \mathbb{R}$ as

$$\begin{aligned}\varphi_X(t) &= \mathbb{E}_X[e^{itX}] = \mathbb{E}_X[\cos(tX) + i\sin(tX)] = \mathbb{E}_X[\cos(tX)] + i\mathbb{E}_X[\sin(tX)] \\ &= \int_{-\infty}^{\infty} e^{itx}\, dF_X(x) = \int_{-\infty}^{\infty} \cos(tx)\, dF_X(x) + i\int_{-\infty}^{\infty} \sin(tx)\, dF_X(x)\end{aligned}$$

using the $dF_X(x)$ notation, where as usual the 'integral' is a sum in the discrete case. For an arbitrary discrete distribution, the pmf takes the form

$$f_X(x) = \sum_{j=1}^{\infty} \omega_j \mathbb{1}_{\{x_j\}}(x)$$

where $x_1, x_2, \ldots$ are a countable collection of (distinct) real values, and $0 \leq \omega_j \leq 1$ and $\sum\limits_{j=1}^{\infty} \omega_j = 1$. We have by direct calculation that

$$\varphi_X(t) = \sum_{j=1}^{\infty} \omega_j \exp\{itx_j\} = \sum_{j=1}^{\infty} \omega_j(\cos(tx_j) + i\sin(tx_j)) = \sum_{j=1}^{\infty} \omega_j \cos(tx_j) + i\sum_{j=1}^{\infty} \omega_j \sin(tx_j).$$

Thus

$$\begin{aligned}|\varphi_X(t)|^2 &= \left(\sum_{j=1}^{\infty} \omega_j \cos(tx_j)\right)^2 + \left(\sum_{j=1}^{\infty} \omega_j \sin(tx_j)\right)^2 \\ &= \sum_{j=1}^{\infty} \omega_j^2 \cos^2(tx_j) + 2\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} \omega_j\omega_k \cos(tx_j)\cos(tx_k) \\ &\quad + \sum_{j=1}^{\infty} \omega_j^2 \sin^2(tx_j) + 2\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} \omega_j\omega_k \sin(tx_j)\sin(tx_k) \\ &= \sum_{j=1}^{\infty} \omega_j^2 + 2\sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} \omega_j\omega_k \cos(t(x_j - x_k))\end{aligned}$$

It can be shown that in this case that $\limsup\limits_{|t|\longrightarrow\infty} |\varphi_X(t)| = 1$; in fact, as $|\varphi_X(t)| \leq 1$, it suffices to show that

$$\limsup_{|t|\longrightarrow\infty} \text{Re } \varphi_X(t) \geq 1.$$

where Re indicates the real part of the complex-valued function. Here

$$\text{Re } \varphi_X(t) = \sum_{j=1}^{\infty} \omega_j \cos(tx_j).$$

Example: $X \sim Poisson(\lambda)$

$$\varphi_X(t) = \exp\{\lambda(e^{it} - 1)\}$$

This function has a real and imaginary part. We have that

$$\varphi_X(t) = \exp\{\lambda(\cos(t) - 1) + i\lambda\sin(t)\} = \exp\{\lambda(\cos(t) - 1)\}(\cos(\lambda\sin(t)) + i\sin(\lambda\sin(t)))$$

so that

$$|\varphi_X(t)| = \exp\{\lambda(\cos(t) - 1)\}.$$

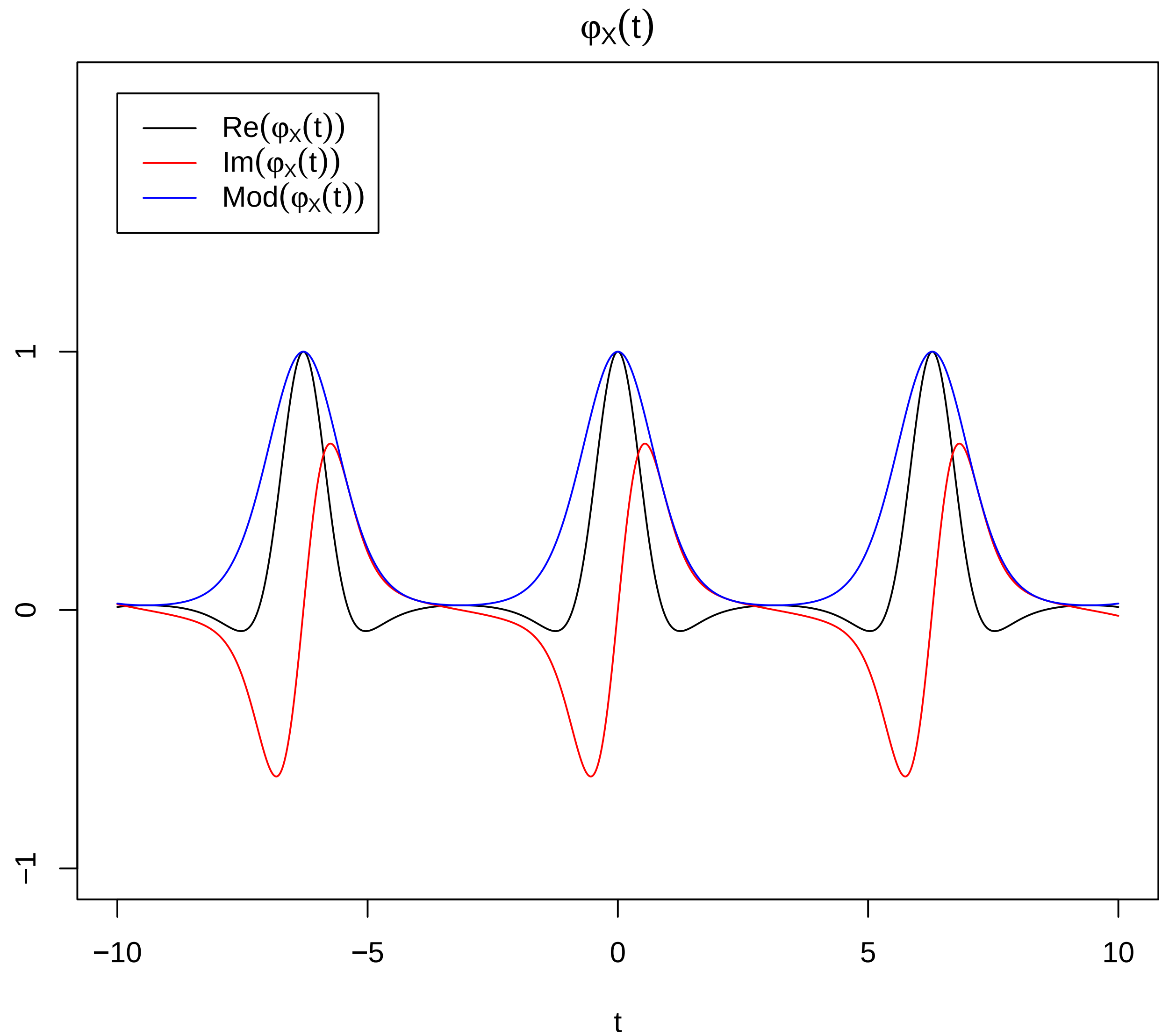
Therefore the function is periodic with period 2π, and

$$\limsup_{|t|\longrightarrow\infty} |\varphi_X(t)| = 1,$$

as is typical for a discrete distribution.

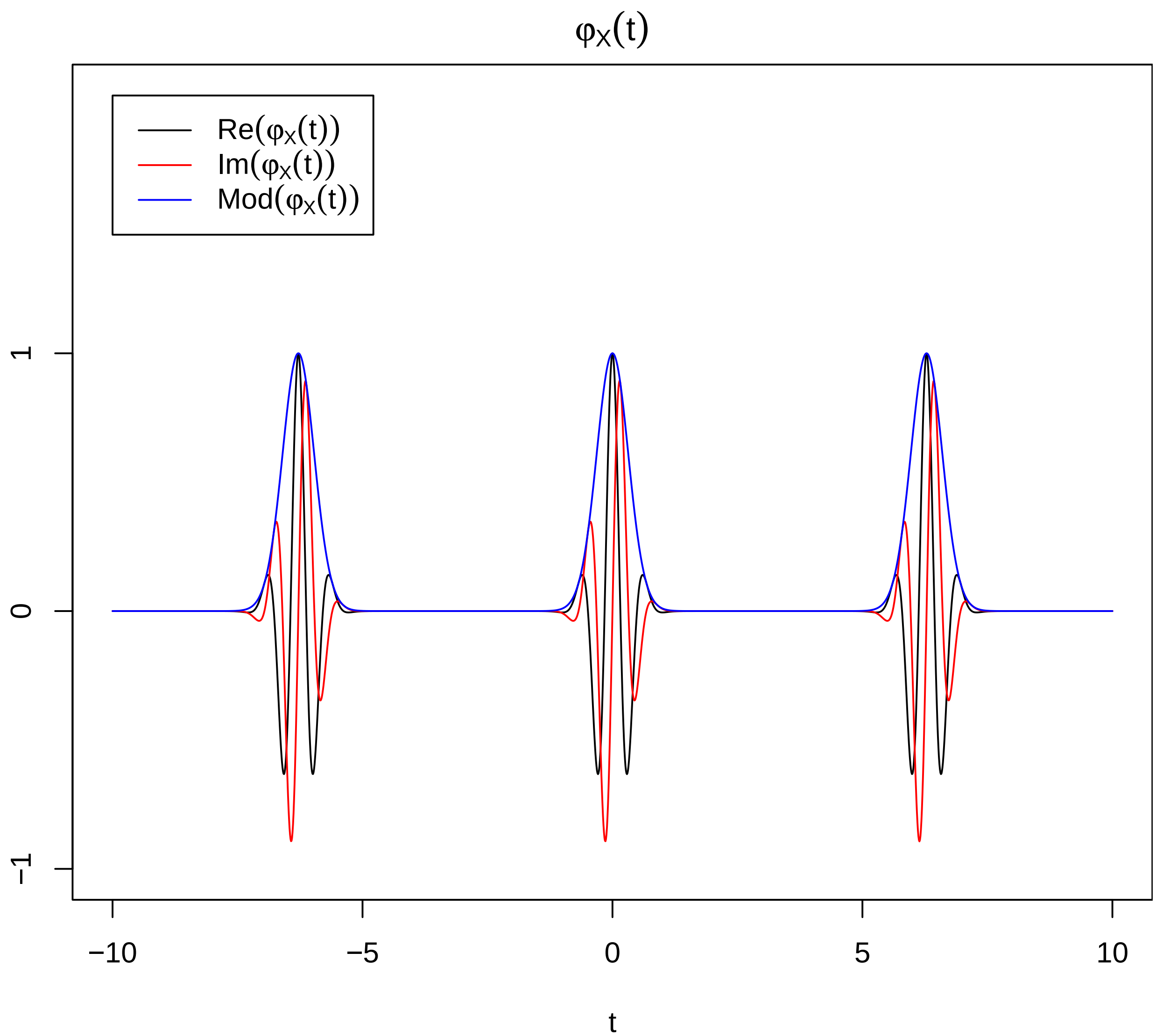
For $\lambda = 2$, we have the following plots

```
tv<-seq(-10,10,by=0.01)
lambda<-2
varphi<-exp(lambda*(exp(1i*tv)-1))
par(mar=c(4,4,2,0))
plot(tv,Re(varphi),type='l',axes=FALSE,
          ylim=range(-1,2),xlab='t',ylab=' ')     #Real part
axis(side=1,at=seq(-10,10,5))
axis(side=2,at=c(-1,0,1))
box()
lines(tv,Im(varphi),col='red')                   #Imaginary part
lines(tv,Mod(varphi),col='blue')                 #Modulus
legend(-10,2,c(expression(Re(varphi[X](t))),
              expression(Im(varphi[X](t))),
              expression(Mod(varphi[X](t)))),col=c('black','red','blue'),lty=1)
title(expression(varphi[X](t)))
```



For $\lambda = 10$:

```
lambda<-10
varphi<-exp(lambda*(exp(1i*tv)-1))
par(mar=c(4,4,2,0))
plot(tv,Re(varphi),type='l',axes=FALSE,
          ylim=range(-1,2),xlab='t',ylab=' ')      #Real part
axis(side=1,at=seq(-10,10,5))
axis(side=2,at=c(-1,0,1))
box()
lines(tv,Im(varphi),col='red')                     #Imaginary part
lines(tv,Mod(varphi),col='blue')                   #Modulus
legend(-10,2,c(expression(Re(varphi[X](t))),
              expression(Im(varphi[X](t))),
              expression(Mod(varphi[X](t)))),col=c('black','red','blue'),lty=1)
title(expression(varphi[X](t)))
```



Example: $X \sim Exponential(\lambda)$

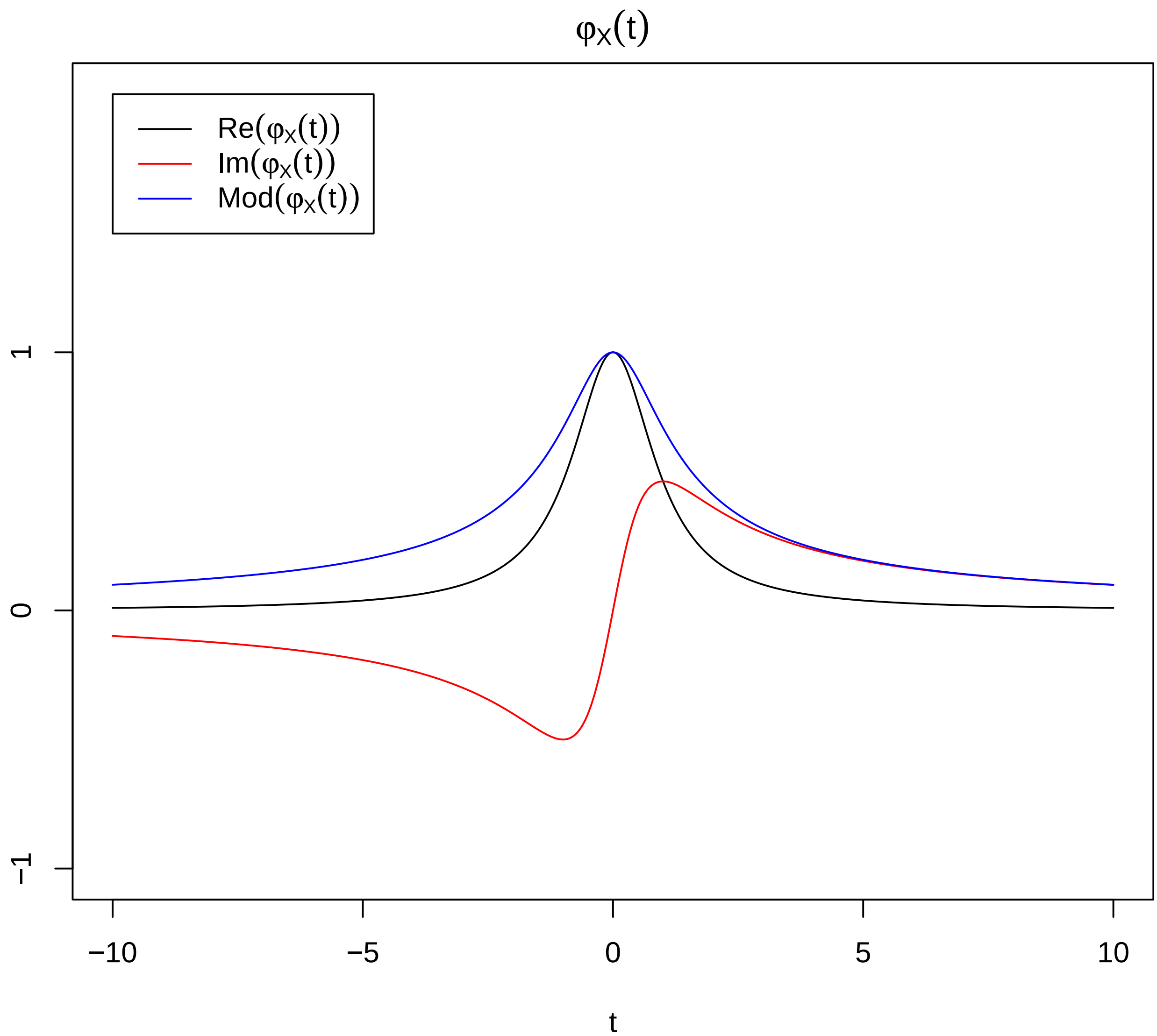
$$\varphi_X(t) = \frac{\lambda}{\lambda - it} = \frac{1}{1 - it\mu} = \frac{1}{1+\mu^2t^2} + i\frac{\mu t}{1+\mu^2t^2}$$

where $\mu = 1/\lambda$. Here

$$|\varphi_X(t)| = \frac{1}{\sqrt{1+\mu^2t^2}} \longrightarrow 0 \quad \text{as } |t| \longrightarrow \infty.$$

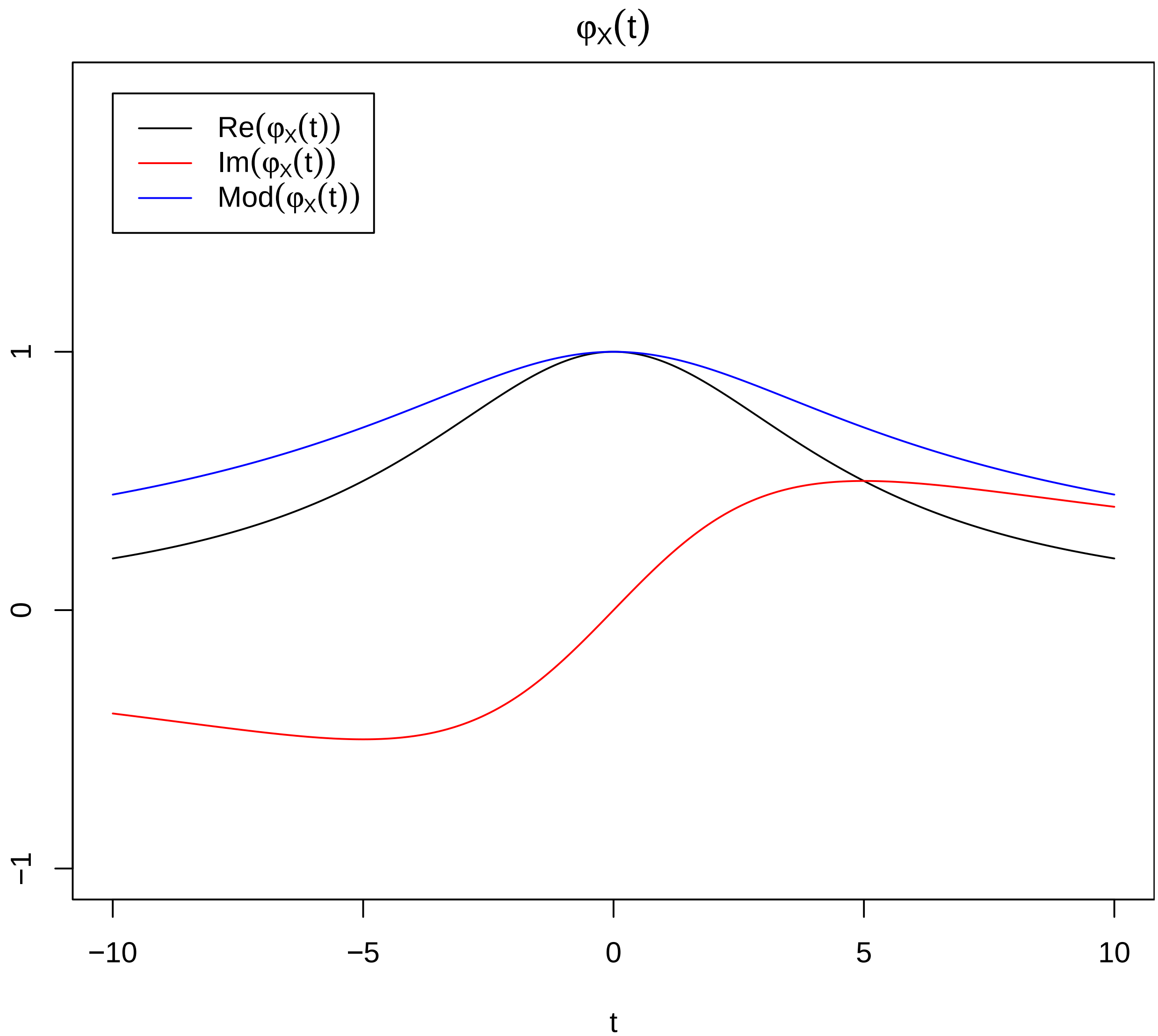
For $\lambda = 1$

```
lambda<-1;mu<-1/lambda
varphi<-1/(1-1i*tv*mu)
par(mar=c(4,4,2,0))
plot(tv,Re(varphi),type='l',axes=FALSE,
          ylim=range(-1,2),xlab='t',ylab=' ')      #Real part
axis(side=1,at=seq(-10,10,5));axis(side=2,at=c(-1,0,1));box()
lines(tv,Im(varphi),col='red')                      #Imaginary part
lines(tv,Mod(varphi),col='blue')                    #Modulus
legend(-10,2,c(expression(Re(varphi[X](t))),
              expression(Im(varphi[X](t))),
              expression(Mod(varphi[X](t)))),col=c('black','red','blue'),lty=1)
title(expression(varphi[X](t)))
```



For $\lambda = 5$

```
lambda<-5;mu<-1/lambda
varphi<-1/(1-1i*tv*mu)
par(mar=c(4,4,2,0))
plot(tv,Re(varphi),type='l',axes=FALSE,
          ylim=range(-1,2),xlab='t',ylab=' ')      #Real part
axis(side=1,at=seq(-10,10,5))
axis(side=2,at=c(-1,0,1));box()
lines(tv,Im(varphi),col='red')                     #Imaginary part
lines(tv,Mod(varphi),col='blue')                   #Modulus
legend(-10,2,c(expression(Re(varphi[X](t))),
              expression(Im(varphi[X](t))),
              expression(Mod(varphi[X](t)))),col=c('black','red','blue'),lty=1)
title(expression(varphi[X](t)))
```

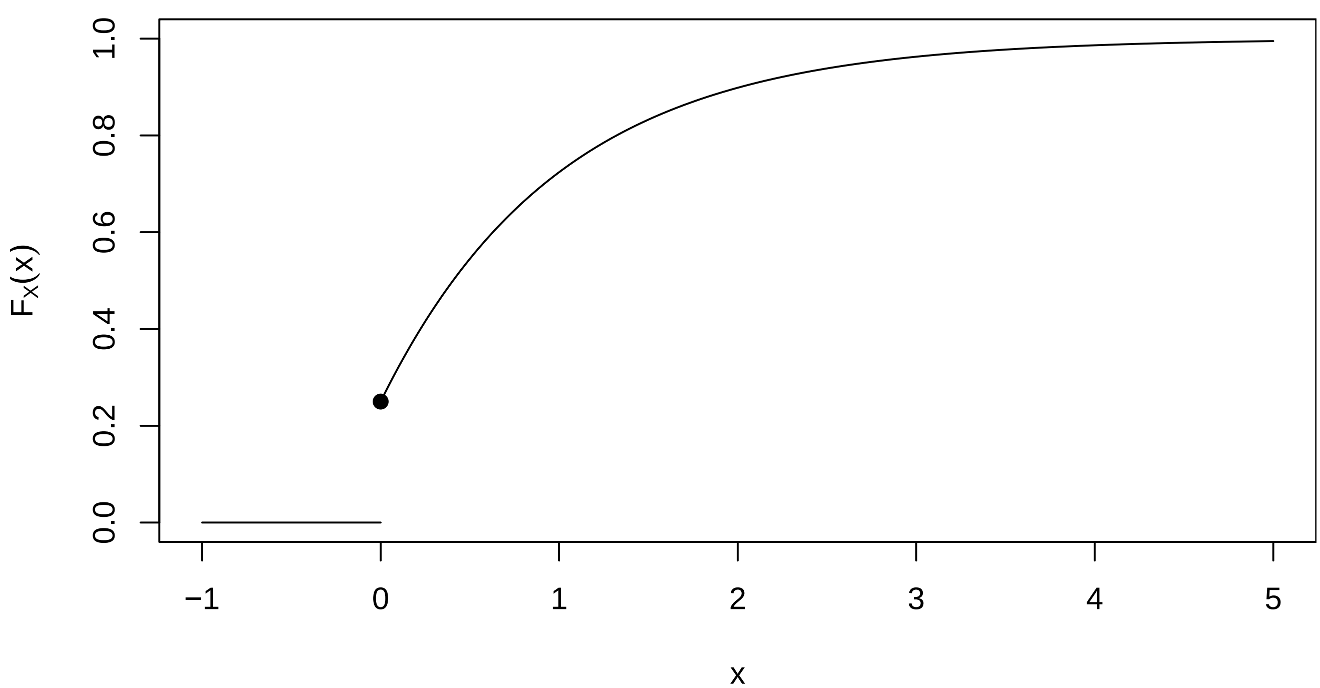


Example: Mixed distribution: suppose

$$F_X(x) = \frac{1}{4}\mathbb{1}_{[0,\infty)}(x) + \frac{3}{4}\mathbb{1}_{(0,\infty)}(x)(1-e^{-x})$$

which places a mass of probability of size 0.25 at $x = 0$, and then adds an $Exponential(1)$ distribution with probability 0.75 to contribute probability for $x > 0$.

```
x<-seq(0,5,by=0.01)
Fx<-0.25*(x>=0)+0.75*(x > 0)*(1-exp(-x))
par(mar=c(4,4,2,0))
plot(x,Fx,type='l',xlim=range(-1,5),ylim=range(0,1),ylab=expression(F[X](x)))
lines(c(-1,0),c(0,0))
points(0,0.25,cex=1.0,pch=19)
```



Then

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} dF_X(x) = \frac{1}{4}e^{it0} + \frac{3}{4}\int_0^{\infty} e^{itx}e^{-x}\, dx = \frac{1}{4} + \frac{3}{4}\frac{1}{1-it}$$

so therefore

$$\varphi_X(t) = \frac{1}{4} + \frac{3}{4}\frac{1}{1+t^2} + i\frac{3}{4}\frac{t}{1+t^2} = \frac{4+t^2}{4(1+t^2)} + i\frac{3t}{4(1+t^2)}$$

Here

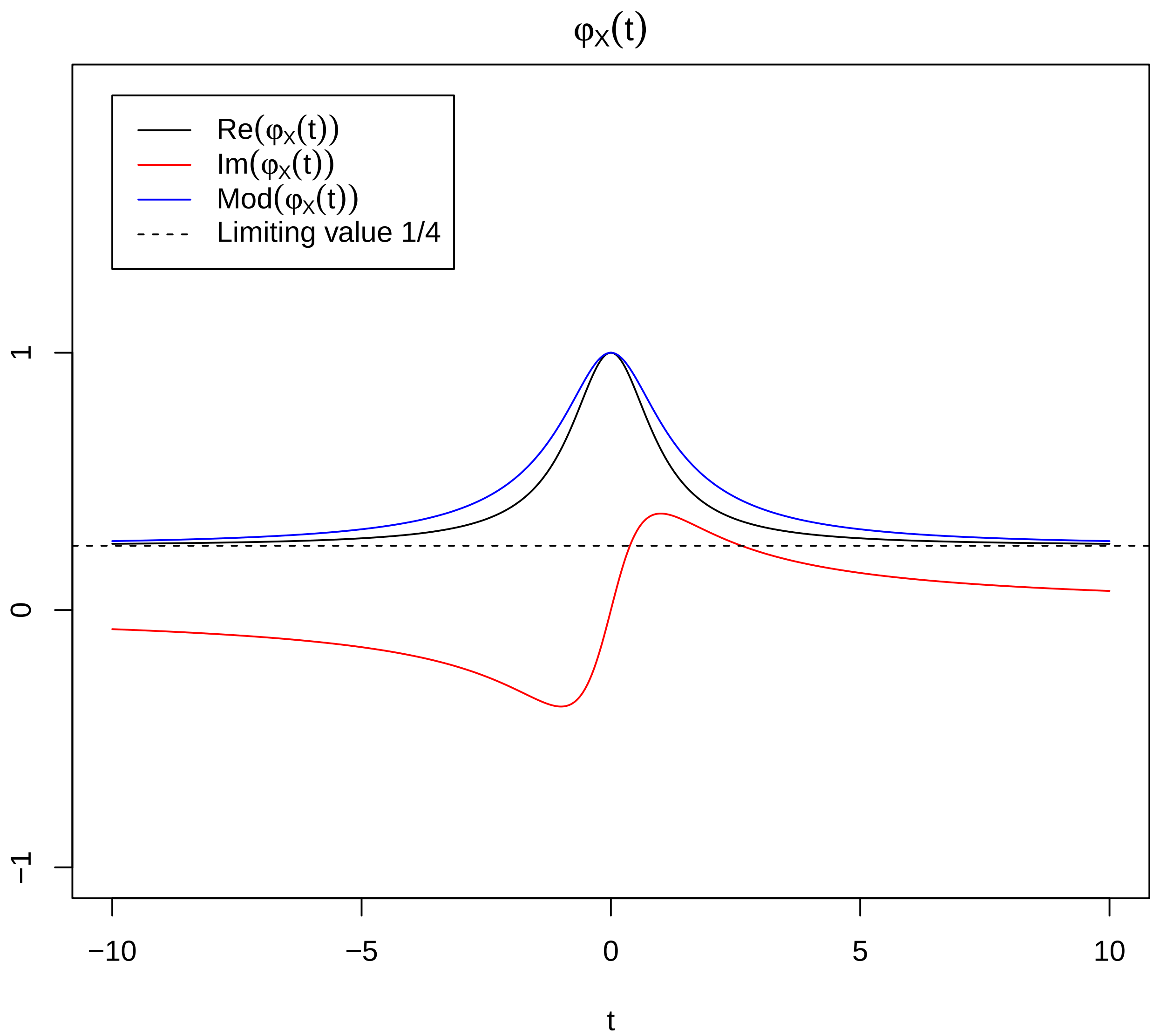
$$|\varphi_X(t)|^2 = \frac{(4+t^2)^2}{16(1+t^2)^2} + \frac{9t^2}{16(1+t^2)^2} = \frac{t^4+17t^2+16}{16(1+t^2)^2} = \frac{t^2+16}{16(1+t^2)}$$

so hence

$$\lim_{|t|\longrightarrow\infty} |\varphi_X(t)| = \frac{1}{4}.$$

Thus the limiting behaviour of this mixed-type distribution is not the same as for the purely discrete or purely (absolutely) continuous type.

```
varphi<-0.25+0.75/(1-1i*tv)
par(mar=c(4,4,2,0))
plot(tv,Re(varphi),type='l',axes=FALSE,
          ylim=range(-1,2),xlab='t',ylab=' ')      #Real part
axis(side=1,at=seq(-10,10,5));axis(side=2,at=c(-1,0,1));box()
lines(tv,Im(varphi),col='red')                     #Imaginary part
lines(tv,Mod(varphi),col='blue')                   #Modulus
legend(-10,2,c(expression(Re(varphi[X](t))),
              expression(Im(varphi[X](t))),
              expression(Mod(varphi[X](t))),
              'Limiting value 1/4'),col=c('black','red','blue'),lty=c(1,1,1,2))
title(expression(varphi[X](t)))
abline(h=0.25,lty=2)
```



Numerical Inversion: We may also attempt numerical inversion of a given cf to get at the pmf or pdf. Suppose

$$\varphi_X(t) = \exp\{-|t|\}$$

which if the cf for the standard Cauchy distribution. Now the relevant inversion formula is

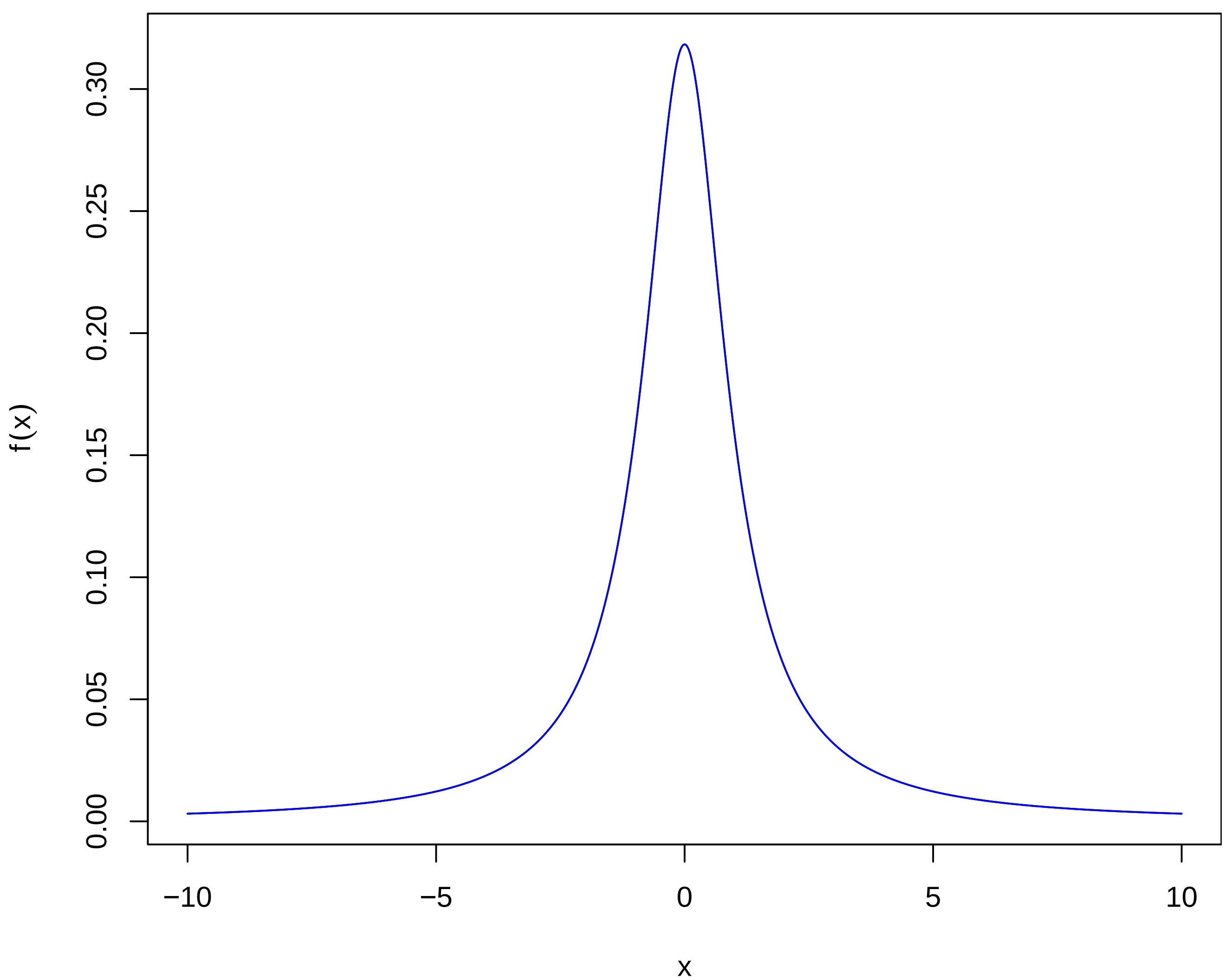
$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi_X(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\exp\{-|t|\}\,dt$$

which we may approximate by

$$\widehat{f}(x) = \frac{1}{2\pi h}\sum_{j=1}^{N} e^{-it_j x}\exp\{-|t_j|\}$$

for a suitable grid $t_1, \ldots, t_N$ with $t_{j+1} - t_j = h$, and N large enough.

```
par(mar=c(4,4,2,0))
x<-seq(-10,10,by=0.01)
fx.hat<-x*0
for(i in 1:length(x)){
    fx.hat[i]<-fx.hat[i]+Re(sum(exp(-1i*tv*x[i])*exp(-abs(tv))))
}
fx.hat<-0.5*fx.hat*(tv[2]-tv[1])/pi
plot(x,fx.hat,type='l',xlab='x',ylab=expression(hat(f)(x)))
lines(x,dt(x,1),col='blue')
```



If

$$\varphi_X(t) = \exp\{-|t|^\alpha\}$$

for $0 < \alpha \leq 2$, the corresponding distribution is still absolutely continuous, but the form of the pdf is not standard. We can use the numerical method.

```
par(mar=c(4,4,2,0))
x<-seq(-10,10,by=0.01)
tv<-seq(-50,50,by=0.01)

alpha<-1.75
fx.hat<-x*0
for(i in 1:length(x)){
    fx.hat[i]<-fx.hat[i]+Re(sum(exp(-1i*tv*x[i])*exp(-abs(tv)^alpha)))
}
fx.hat<-0.5*fx.hat*(tv[2]-tv[1])/pi
plot(x,fx.hat,type='l',xlab='x',ylab=expression(hat(f)(x)))
```