MATH 556 – SOLUTIONS TO THE FINAL EXAM 2011

PROBLEM 1: (17 marks)

(a) Set
$$g : (0,1) \longrightarrow \mathbb{R}$$
$$u \longmapsto \mu - \beta \ln(-\ln(u)).$$

(1) $$g^{-1} : x \longmapsto e^{-e^{-\frac{x-\mu}{\beta}}}$$

$$(g^{-1})'(x) \overset{(1)}{=} e^{-e^{-\frac{x-\mu}{\beta}}} \cdot e^{-\frac{x-\mu}{\beta}} \cdot \frac{1}{\beta}$$

g is monotone increasing, g^{-1} cont. differentiable $\Big\}$ and hence X has a density

$$f_X(x) \overset{(1)}{=} f_U(g^{-1}(u))\ |(g^{-1})'(u)|$$

$$\overset{(1)}{=} \underbrace{1\left(e^{-e^{-\frac{x-\mu}{\beta}}} \in (0,1)\right)}_{=1 \text{ because } e^{-\frac{x-\mu}{\beta}} \in (0,\infty).} e^{-e^{-\frac{x-\mu}{\beta}}} \cdot e^{-\frac{x-\mu}{\beta}} \cdot \frac{1}{\beta}$$

for $\overset{(1)}{x \in \mathbb{R}}$.

(b) $$M_X(t) \overset{(1)}{=} E\, e^{tX} = \int_{-\infty}^{\infty} e^{tx}\, e^{-e^{-\frac{x-\mu}{\beta}}}\, e^{-\frac{x-\mu}{\beta}} \cdot \frac{1}{\beta}\, dx$$

Set $y = e^{-\frac{x-\mu}{\beta}}$. Then $dy = -\frac{1}{\beta} e^{-\frac{x-\mu}{\beta}}$

and $x = \mu - \beta \ln y$. Thus

$$M_X(t) \overset{1}{=} \int_0^\infty e^{t(\mu - \beta \ln y)} e^{-y} \, dy$$

$$= e^{t\mu} \int_0^\infty y^{1-\beta t - 1} e^{-y} \, dy$$

$$\overset{1}{=} e^{t\mu} \, \Gamma(1-\beta t) \quad \text{for} \quad 1-\beta t > 0 \iff t < \frac{1}{\beta}. \quad 1$$

(c) $E(Y) \overset{1}{=} M'_Y(0)$.

$$S'_Y(t) \overset{1}{=} \frac{M'_Y(t)}{M_Y(t)} \Rightarrow S'_Y(0) = \frac{EY}{M_Y(0)} = EY.$$

$$S''_Y(t) \overset{1}{=} \frac{M_Y(t) M''_Y(t) - \{M'_Y(t)\}^2}{M_Y^2(t)} \underset{t=0}{\overset{1}{=}} \frac{EY^2 - (EY)^2}{1} = \operatorname{var} Y.$$

(d)

$$S_X(t) = \ln(M_X(t)) = t\mu + \ln\Gamma(1-\beta t)$$

$$S'_X(t) \overset{1}{=} \mu + \frac{\Gamma'(1-\beta t)}{\Gamma(1-\beta t)}(-\beta)$$

$$S'_X(0) \overset{1}{=} \mu + \gamma\beta.$$

$$S''_X(t) \overset{1}{=} (-\beta) \frac{\Gamma''(1-\beta t)(-\beta)\Gamma(1-\beta t) + \beta\{\Gamma'(1-\beta t)\}^2}{\Gamma^2(1-\beta t)}$$

$$S''_X(0) = (-\beta)\,\frac{(\frac{\pi^2}{6}+\gamma^2)(-\beta) + \beta\cdot\gamma^2}{1}$$

$$\overset{1}{=} \beta^2\,\frac{\pi^2}{6}.$$

PROBLEM 2 (16 marks)

(a) The transformation to polar coordinates is

$$T:\ \mathbb{R}^2 \longrightarrow \mathbb{R}^+ \times (0, 2\pi)$$
$$(x,y) \longmapsto \left(\sqrt{x^2+y^2},\ \arctan\frac{y}{x}\right)$$

$$T^{-1}:\ [0,\infty)\times(0,2\pi) \longrightarrow \mathbb{R}^2$$
$$\text{(1)}\quad (r,\theta) \longmapsto (r\cdot\cos\theta,\ r\sin\theta)$$

$$J \overset{1}{=} \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \rightsquigarrow |J| \overset{1}{=} r$$

Therefore, the joint density of (R, θ) is

$$f_{(R,\theta)}(r,\theta) = \frac{1}{2\pi}\, e^{-\frac{r^2\cos^2\theta}{2}}\, e^{-\frac{r^2\sin^2\theta}{2}}\, r$$

$$\overset{1}{=} \frac{1}{2\pi}\cdot e^{-\frac{r^2}{2}}\cdot r, \quad r\in(0,\infty),\ \theta\in(0,2\pi)$$

(b) This means that (5) $R \perp \theta$ and $R \overset{1}{\sim} r\cdot e^{-\frac{r^2}{2}}\,1(0,\infty)$.

(d) Consider the transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longrightarrow \left(\frac{x}{y}, y\right)$$

$$T^{-1}(u,v) \longmapsto (uv, v) \quad ; \quad J = \begin{pmatrix} v & 0 \\ u & 1 \end{pmatrix}$$

(1) $$\rightsquigarrow |J| = v.$$

The density of $\left(\frac{X}{Y}, Y\right)$ is

(1) $$f_{\left(\frac{X}{Y}, Y\right)}(u,v) = \frac{1}{2\pi} e^{-\frac{u^2 v^2}{2}} e^{-\frac{v^2}{2}} |v| \ , \ u, v \in \mathbb{R}$$

Therefore, $$f_{\frac{X}{Y}}(u) \overset{(1)}{=} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{v^2(u^2+1)}{2}} |v| \, dv$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-\frac{v^2(u^2+1)}{2}} \cdot v \, dv$$

$$\overset{(1)}{=} \frac{1}{\pi(1+u^2)} \int_0^{\infty} e^{-t} \, dt$$

$$= \frac{1}{\pi(1+u^2)}$$

Hence, $\frac{X}{Y} \sim$ Cauchy $(0,1)$, so MGF does not

(2) exist because the mean is infinite.

PROBLEM 3

(a) $$f(x \mid \chi, \psi) = \frac{\exp\sqrt{\chi\psi}}{\sqrt{2\pi x^2}} \sqrt{\chi} \cdot e^{-\frac{1}{2}x^{-1}\chi - \frac{1}{2}\psi x} \mathbb{1}(x>0)$$

Set

$$h(x) = \mathbb{1}(x>0) \cdot \frac{1}{\sqrt{2\pi x^2}}$$

1 $$c(\chi, \psi) = \exp\sqrt{\chi\psi} \cdot \sqrt{\chi}$$

$$t_1(x) = -\frac{1}{2x}, \quad t_2(x) = -\frac{1}{2}x$$

1 $$w_1(\chi, \psi) = \chi, \quad w_2(\chi, \psi) = \psi.$$

1 $$\eta_1 = \chi, \quad \eta_2 = \psi.$$

$$\mathcal{H} = \left\{ (\eta_1, \eta_2) : \int_0^\infty \frac{1}{\sqrt{2\pi x^2}} e^{-\eta_1 \frac{1}{2x}} e^{-\eta_2 \frac{1}{2} \cdot x} dx < \infty \right\}$$

1 $$= \{ (\eta_1, \eta_2) : \eta_1 \in (0, \infty), \eta_2 \in (0, \infty) \}.$$

(b) Score function is

$$S_i(x) = \frac{\partial}{\partial \theta_i} \log f(x \mid \theta)$$

under suitable regularity conditions,

$$E(S_i(X)) = 0$$

this is because

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial \theta_i} \log f(x|\theta)\, f(x|\theta)\, dx$$

$$① \quad = \int_{-\infty}^{\infty} \frac{\partial}{\partial \theta_i} f(x|\theta)\, dx = \frac{\partial}{\partial \theta_i} \int f(x|\theta)\, dx = 0.$$

In the canonical parametrization,

$$f(x|\eta) = h(x)\, c(\eta)\, e^{\sum_{i=1}^{k} t_i(x)\eta_i}$$

$$\frac{\partial \log f(x|\eta)}{\partial \eta_i} = \frac{\partial}{\partial \eta_i}\left(\log h(x) + \log c(\eta) + \sum_{i=1}^{k} t_i(x)\eta_i \right)$$

$$= \frac{\partial}{\partial \eta_i} \log c(\eta) + t_i(x).$$

$$\Rightarrow \quad E(t_i(X)) \underset{⑦}{=} -\frac{\partial}{\partial \eta_i} \log c(\eta).$$

$$\operatorname{cov}(S_i(X), S_j(X)) = \int_{-\infty}^{\infty} \frac{\partial}{\partial \theta_i} \log f(x|\theta) \frac{\partial}{\partial \theta_j} \log f(x|\theta)\, f(x|\theta)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial \theta_i} f(x|\theta) \frac{\partial}{\partial \theta_j} f(x|\theta) \frac{1}{f(x|\theta)}\, dx$$

on the other hand,

$$\int_{-\infty}^{\infty} \frac{\partial^2}{\partial \theta_i \partial \theta_j} \log f\, f(x|\theta)\, dx$$

$$① \quad = \int_{-\infty}^{\infty} \frac{\frac{\partial^2}{\partial \theta_i \partial \theta_j} f(x|\theta) \cdot f(x|\theta) - \frac{\partial}{\partial \theta_i} f \frac{\partial}{\partial \theta_j} f}{f^2(x|\theta)}\, f(x|\theta)\, dx$$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{\partial^2}{\partial\theta_i\partial\theta_j} f(x|\theta)\, dx}_{0} - \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta_i} f \frac{\partial}{\partial\theta_j} f \cdot \frac{1}{f}\, dx$$

$$\Rightarrow \operatorname{cov}(S_i(X), S_j(X)) = -\int_{-\infty}^{\infty} \frac{\partial^2}{\partial\theta_i\partial\theta_j} \log f(x|\theta) \cdot f(x|\theta)\, d$$

In the canonical parametrization,

$$\operatorname{cov}(S_i(X), S_j(X)) = \operatorname{cov}(t_i(X), t_j(X))$$

$$\overset{(1)}{=} -\frac{\partial^2}{\partial\eta_i\partial\eta_j} \log c(\eta).$$

(c) $$E\left(-\frac{1}{2}\frac{1}{X}\right) \overset{(1)}{=} -\frac{\partial}{\partial\chi} \cdot \log\left(e^{\sqrt{\chi\psi}} \cdot \sqrt{\chi}\right)$$

$$\cancel{= -\frac{\partial}{\partial\psi}\sqrt{\chi\psi} = -\frac{1}{2}\sqrt{\psi}\cdot\frac{1}{\sqrt{\chi}}}$$

$$= -\frac{\partial}{\partial\chi}\left(\sqrt{\chi\psi} + \frac{1}{2}\log\chi\right)$$

$$= -\sqrt{\chi\psi}\,\frac{1}{2}\frac{1}{\sqrt{\chi}} + \frac{1}{2}\frac{1}{\chi}$$

$$= -\frac{1}{2}\left(\frac{\sqrt{\psi}}{\sqrt{\chi}} + \frac{1}{\chi}\right).$$

$$\Rightarrow E\left(\frac{1}{X}\right) \overset{(1)}{=} \frac{\sqrt{\psi}}{\sqrt{\chi}} + \frac{1}{\chi}.$$

$$\operatorname{var}\left(-\frac{1}{2}\frac{1}{X}\right) \overset{(1)}{=} \frac{\partial}{\partial\chi}\left(-\frac{1}{2}\left(\frac{\sqrt{\psi}}{\sqrt{\chi}} + \frac{1}{\chi}\right)\right)$$

$$= -\frac{1}{2}\sqrt{\psi}\left(-\frac{1}{2}\cdot\frac{1}{\chi^{\frac{3}{2}}} + 1\,\frac{1}{\chi^2}\right)$$

$$= \frac{1}{4}\frac{\sqrt{\psi}}{\chi^{3/2}} + \frac{1}{2}\frac{1}{\chi^2}$$

$$\Rightarrow \operatorname{var}\left(\frac{1}{X}\right) \overset{1}{=} \frac{\sqrt{\psi}}{\sqrt{x}\cdot x} + \frac{2}{x^2}.$$

(d) $$\operatorname{cov}\left(-\frac{1}{2}X, -\frac{1}{2}\frac{1}{X}\right) \overset{1}{=} \frac{1}{4}\operatorname{cov}\left(X, \frac{1}{X}\right)$$

$$\overset{1}{=} \frac{\partial}{\partial \psi}\left(-\frac{1}{2}\frac{\sqrt{\psi}}{\sqrt{x}} - \frac{1}{2}\frac{1}{x}\right)$$

$$\overset{1}{=} -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x\psi}}$$

$$\Rightarrow \operatorname{cov}\left(X, \frac{1}{X}\right) \overset{1}{=} -\frac{1}{\sqrt{x\psi}}$$

(e)
- not always well defined 1
- $\operatorname{cov}(X,Y) = 0 \not\Rightarrow X \perp Y$ 1
- may not attain the bounds ± 1. 1

PROBLEM 4

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|M=m}(x)\cdot f_M(m)\,dm$$

$$\overset{1}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi}}\frac{1}{\tau} e^{-\frac{(m-\mu)^2}{2\tau^2}}\,dm$$

$$e^{-\frac{(x-m)^2}{2\sigma^2}}\cdot e^{-\frac{(m-\mu)^2}{2\tau^2}} = \cancel{e^{-\frac{x^2\tau^2 - 2mx\tau^2 + m^2\tau^2 + m^2\sigma^2 - 2m\mu\sigma^2 + \mu^2\sigma^2}{2\sigma^2\tau^2}}}$$

$$\cancel{= e^{-\frac{\tau^2(x-m)^2 + \sigma^2(m-\mu)^2}{2\sigma^2\tau^2}}} =$$

$$= e^{-\frac{m^2(\tau^2+\sigma^2) - 2m(\tau^2 x + \mu\sigma^2) + \tau^2 x^2 + \sigma^2\mu^2}{2\sigma^2\tau^2}}$$

$$\overset{①}{=} e^{-\frac{\left(m - \frac{\tau^2 x + \mu\sigma^2}{\tau^2+\sigma^2}\right)^2}{2\sigma^2\tau^2 / (\tau^2+\sigma^2)}} \; e^{\frac{(\tau^2 x + \mu\sigma^2)^2}{2\sigma^2\tau^2(\tau^2+\sigma^2)} - \frac{\tau^2 x^2 + \sigma^2\mu^2}{2\sigma^2\tau^2}}$$

Hence,

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\tau^2+\sigma^2}} \, e^{\frac{\cancel{\tau^4 x^2} + \cancel{\mu^2\sigma^4} + 2x\mu\sigma^2\sigma^2 - \cancel{\tau^4 x^2} - \tau^2\sigma^2 x^2 - \cancel{\sigma^4\mu^2} - \sigma}{2\tau^2\sigma^2(\tau^2+\sigma^2)}}$$

$$\times \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\tau^2+\sigma^2}{1}} \frac{1}{\tau\sigma} e^{-\frac{\left(m - \frac{\tau^2 x + \mu\sigma^2}{\tau^2+\sigma^2}\right)^2}{2\tau^2\sigma^2/(\tau^2+\sigma^2)}} \, dm}_{① \; = 1}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\tau^2+\sigma^2}} e^{-\frac{(x-\mu)^2}{2(\tau^2+\sigma^2)}}$$

①

$$\Rightarrow X \sim N(\mu, \tau^2+\sigma^2).$$

(b) RIGHT-HAND side is

$$E\left(E(XY|Z) \overset{①}{-} E(X|Z)E(Y|Z)\right)$$

$$+ E\left(E(X|Z)E(Y|Z)\right) \overset{①}{-} E(E(X|Z))\,E(E(Y|Z))$$

$$\overset{①}{=} E(XY) - EX\,EY \overset{①}{=} \operatorname{cov}(X,Y).$$

(c) $\operatorname{cov}(X_1, X_2) \overset{1}{=} E(\operatorname{cov}(X_1, X_2 \mid M))$

$+ \operatorname{cov}(E(X_1 \mid M), E(X_2 \mid M))$

$\overset{1}{=} 0 + \operatorname{cov}(M, M) = \operatorname{var}(M) \overset{1}{=} \tau^2.$

$\operatorname{var}(X_1) \overset{1}{=} \tau^2 + \sigma^2 = \operatorname{var}(X_2).$

$\Rightarrow \operatorname{corr}(X_1, X_2) = \dfrac{\tau^2}{\tau^2 + \sigma^2} > 0.$

(d) no, because corr > 0. 4

PROBLEM 5

a) MGF of $n\overline{X}_n$:

$E\left(e^{\left(\frac{t}{\cancel{n}} X_1 + \cdots + \frac{t}{\cancel{n}} X_n\right)}\right) \overset{1}{=} \prod_{i=1}^{n} M_{X_i}\left(\frac{t}{\cancel{n}}\right)$

$\overset{1}{=} e^{\lambda(e^t - 1)\cdot n} \sim X_1 + \cdots + X_n \overset{1}{\sim} \operatorname{Poi}(n \cdot \lambda).$

Hence, $\overline{X}_n$ has support $\{0, \frac{1}{n}, \frac{2}{n}, \ldots\}$ and

$P\left(\overline{X}_n = \frac{k}{n}\right) = P(X_1 + \cdots + X_n = k) \overset{1}{=} \dfrac{(n\lambda)^k}{k!} e^{-n\lambda}.$

(b) $\operatorname{var}(X_1(X_1 - 1)) = E(X_1^2 (X_1 - 1)^2) - \{E(X_1(X_1 - 1))\}^2$

$= E(X_1^4 - 2X_1^3 + X_1^2) - \{E(X_1^2) - E(X_1)\}^2$

$\overset{1}{=} EX_1^4 - 2EX_1^3 + EX_1^2 - \{E(X_1^2) - E(X_1)\}^2$

$$M_{X_1}(t) = e^{\lambda(e^t-1)}$$

① $\begin{cases} M'_{X_1}(t) = e^{\lambda(e^t-1)} \cdot \lambda e^t \rightsquigarrow EX_1 = \lambda \\ M''_{X_1}(t) = e^{\lambda(e^t-1)}\left(\lambda^2 e^{2t} + \lambda e^t\right) \rightsquigarrow EX_1^2 = \lambda^2 + \lambda \end{cases}$

$$M'''_{X_1}(t) = e^{\lambda(e^t-1)}\left(\lambda^3 e^{3t} + \lambda^2 e^{2t} + 2\lambda^2 e^{2t} + \lambda e^t\right)$$

① $$\rightsquigarrow EX_1^3 = \lambda^3 + 3\lambda^2 + \lambda$$

$$M^{IV}_{X_1}(t) = e^{\lambda(e^t-1)}\left(\lambda^4 e^{4t} + 3\lambda^3 e^{3t} + \lambda^2 e^{2t} + 3\lambda^3 e^{3t} + 6\lambda^2 e^{2t} + \lambda e^t\right)$$

$$\rightsquigarrow EX_1^4 = \lambda^4 + 3\lambda^3 + \lambda^2 + 3\lambda^3 + 6\lambda^2 + \lambda$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

$$\mathrm{var}(X_1(X_1-1)) = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda - 2\lambda^3 - 6\lambda^2 - 2\lambda + \lambda^2 + \lambda - (\lambda^2 + \lambda - \lambda)^2$$

$$\overset{①}{=} 4\lambda^3 + 2\lambda^2$$

(c) $g(x) = \sqrt{x}$ is concave

$-g(x)$ is convex.

Jensen's inequality: $-g(EX) \overset{②}{\leq} E(-g(X))$

$$E\left(-\sqrt{\tfrac{1}{n}\textstyle\sum_i X_i(X_i-1)}\right) \geq -\sqrt{E\left(\tfrac{1}{n}\textstyle\sum_i X_i(X_i-1)\right)}.$$

$$E(\sqrt{\cdots}) \underset{①}{\leq} \sqrt{E(X_1(X_1-1))} \underset{①}{=} \sqrt{\lambda^2} = \lambda.$$

(d) $Z_i = X_i(X_i - 1)$ are iid with $E Z_1 = \lambda^2$ (2)

$\operatorname{var}(Z_i) < \infty$.

WLLN: $\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow{P} \lambda^2$ (1)

CMT: $\sqrt{\frac{1}{n}\sum_{i=1}^{n} Z_i} \xrightarrow{P} \lambda.$ (1)

(e) For $y < 0$, $P(Y_n \le y) = 0$.

For $y \ge 0$, $P(Y_n \le y) = P\left(\frac{1}{n}\sum_{i=1}^{n} (X_i)(X_i - 1) \le y^2\right)$ (1)

By the clt, $\frac{\bar{Z}_n - \lambda^2}{\sqrt{4\lambda^3 + 2\lambda^2}} \cdot \sqrt{n} \rightsquigarrow N(0,1)$. (1) Hence

$$P(\bar{Z}_n \le y^2) = P\left(\frac{\bar{Z}_n - \lambda^2}{\sqrt{4\lambda^3 + 2\lambda^2}}\sqrt{n} \le \sqrt{n}\,\frac{y^2 - \lambda^2}{\sqrt{}}\right) \text{ (1)}$$

$$\approx \Phi\left(\frac{y^2 - \lambda^2}{\sqrt{4\lambda^3 + 2\lambda^2}} \cdot \sqrt{n}\right). \text{ (1)}$$

PROBLEM 6

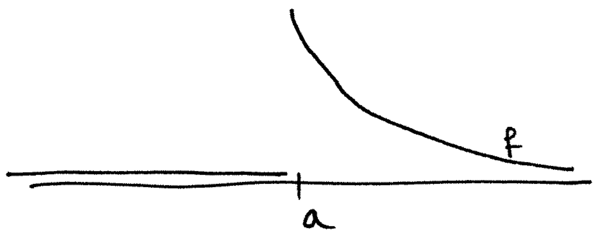
a) $P(Y_n \le y) = 1 - P(Y_n > y)$

$\overset{(1)}{=} 1 - P(X_1 > y, \ldots, X_n > y)$

$\overset{(1)}{=} 1 - \prod_{i=1}^{n} P(X_i > y)$

$\overset{(1)}{=} 1 - (1 - F(y))^n$.

(b)



$$P\left(n(Y_n - a) \le y\right) \overset{1}{=} P\left(Y_n \le \frac{y}{n} + a\right)$$

$$\overset{1}{=} 1 - \left(1 - F\left(\frac{y}{n} + a\right)\right)^n$$

$$\overset{1}{=} 1 - \left(1 - \frac{F\left(\frac{y}{n} + a\right) \cdot n}{n}\right)^n$$

$$n \cdot F\left(\frac{y}{n} + a\right) = \frac{F\left(\frac{y}{n} + a\right) - F(a)}{\frac{y}{n}} \cdot y$$

$$\overset{1}{\longrightarrow} f(a) \cdot y$$

Hence, $$P\left(n(Y_n - a) \le y\right) \underset{1}{\longrightarrow} \begin{cases} 1 - e^{-f(a) \cdot y}, & y > 0 \\ 0, & y < 0 \end{cases}$$

(c) $$\underbrace{n \cdot (Y_n - a)}_{\rightsquigarrow \text{Exp}(f(a))} \cdot \underbrace{\frac{1}{n}}_{\to 0} \xrightarrow{P} 0$$ by Slutsky's Lemma. 4