The characteristic function for a random variable $X$ with pmf/pdf $f_X$ is defined for $t \in \mathbb{R}$ as

$$
\phi_X(t) = \mathbb{E}[e^{itX}] = \mathbb{E}[\cos(tX) + i\sin(tX)] = \mathbb{E}[\cos(tX)] + i\mathbb{E}[\sin(tX)].
$$

In general $\phi_X(t)$ is a complex-valued function. If $X$ is discrete, taking values on $\mathbb{X} = \{x_1, x_2, \ldots\}$

$$
\mathbb{E}[\cos(tX)] = \sum_{j=1}^{\infty} \cos(tx_j)f_X(x_j)
$$

$$
\mathbb{E}[\sin(tX)] = \sum_{j=1}^{\infty} \sin(tx_j)f_X(x_j)
$$

Now,

$$
\sum_{j=1}^{\infty} \cos(tx_j)f_X(x_j) \leq \sum_{j=1}^{\infty} |\cos(tx_j)f_X(x_j)| \leq \sum_{j=1}^{\infty} f_X(x_j) = 1
$$

with a similar result for $\sin$, so the two expectations are finite, so $\phi_X(t)$ exists. The same argument works for $X$ continuous, where

$$
\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x) \, dx = \int_{-\infty}^{\infty} \cos(tx)f_X(x) \, dx + i \int_{-\infty}^{\infty} \sin(tx)f_X(x) \, dx
$$

**Example Double-Exponential (or Laplace) distribution**

$$
f_X(x) = \frac{1}{2} e^{-|x|} \quad x \in \mathbb{R}
$$

Then

$$
\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{2} e^{-|x|} \, dx = \int_{0}^{\infty} \cos(tx)e^{-x} \, dx + i \int_{0}^{\infty} \sin(tx)e^{-x} \, dx.
$$

Integrating by parts we have

$$
\phi_X(t) = [-\cos(tx)e^{-x}]_0^{\infty} + \int_{0}^{\infty} t\sin(tx)e^{-x} \, dx
$$

$$
= 1 + [-t\sin(tx)e^{-x}]_0^{\infty} - \int_{0}^{\infty} t^2\cos(tx)e^{-x} \, dx
$$

$$
= 1 - t^2 \phi_X(t)
$$

Therefore

$$
\phi_X(t) = \frac{1}{1 + t^2}
$$
**Example Normal distribution**

\[ f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R} \]

Then

\[ \varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx. \]

Completing the square

\[ \varphi_X(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-it^2/2} dx. \]

Therefore

\[ \varphi_X(t) = e^{-t^2/2}. \]

The following results also hold:

- \( \varphi_X(t) \) is **continuous** for all \( t \); this follows as \( \cos \) and \( \sin \) are continuous functions of \( x \), and sums and integrals of continuous functions are also continuous. In fact, we can prove the stronger result that \( \varphi_X(t) \) is **uniformly continuous** on \( \mathbb{R} \).
- \( \varphi_X(t) \) is **bounded in modulus** by 1, as

  \[ |\varphi_X(t)| \leq \mathbb{E}[|e^{itX}|] = \mathbb{E}[1] = 1 \]

- The derivatives of \( \varphi_X(t) \) are not guaranteed to be finite; we can consider

  \[ \varphi_X^{(r)}(t) = \frac{d^r}{dt^r} \{ \varphi_X(t) \} \]

  but this quantity may not be defined, or finite, at any given \( t \); if \( r = 1 \)

  \[ \varphi_X^{(1)}(t) = \mathbb{E}[X \cos(tX)] + i \mathbb{E}[X \sin(tX)]. \]

  but there is no guarantee that either expectation is finite. For example, for the Cauchy distribution

  \[ \varphi_X(t) = e^{-|t|} \]

  which undefined derivative at \( t = 0 \).

**Inversion Formula**

A general inversion formula in 1-D gives the method via which \( f_X \) or \( F_X \) can be computed from \( \varphi_X \).

- Let \( \overline{F}_X(x) \) be defined by

  \[ \overline{F}_X(x) = \frac{1}{2} \left\{ F_X(x) + \lim_{y \to x^-} F_X(y) \right\}. \]

  Then for \( a < b \)

  \[ \overline{F}_X(b) - \overline{F}_X(a) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \left( \frac{e^{-iat} - e^{-ibt}}{it} \right) \varphi_X(t) \, dt \]

- For an alternative statement, let \( a \) and \( a + h \) for \( h > 0 \) be continuity points of \( F_X \). Then

  \[ F_X(a + h) - F_X(a) = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \left( \frac{1 - e^{-ith}}{it} \right) e^{-ita} \varphi_X(t) \, dt \]
In certain circumstances we may compute \( f_X \) from \( \varphi_X \) more straightforwardly.

(I) If \( X \) is discrete taking values on the integers. Then

\[
\varphi_X(t) = \sum_{x=-\infty}^{\infty} e^{itx} f_X(x).
\]

For integer \( j \)

\[
\int_{-\pi}^{\pi} e^{i(j-x)t} \, dt = \begin{cases} 
2\pi & \text{if } x = j \\
0 & \text{if } x \neq j
\end{cases}
\]

Thus for any fixed \( x \)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ixt} \varphi_X(t) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ixt} \left( \sum_{j=-\infty}^{\infty} e^{itj} f_X(j) \right) \, dt = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \left( \int_{-\pi}^{\pi} e^{i(j-x)t} \, dt \right) f_X(j) = f_X(x)
\]

(as only the term when \( j = x \) is non-zero in the sum) so we have the inversion formula: for \( x \in \mathbb{Z} \)

\[
f_X(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ixt} \varphi_X(t) \, dt.
\]

(II) If \( X \) is continuous and absolutely integrable

\[
\int_{-\infty}^{\infty} |\varphi_X(t)| \, dt < \infty
\]

then

\[
f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \varphi_X(t) \, dt
\]

**Example** Suppose that for \( t \in \mathbb{R} \),

\[
\varphi_X(t) = e^{-|t|}.
\]

Clearly this function is absolutely integrable, so we have

\[
f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-|t|} \, dt = \frac{1}{\pi} \int_{0}^{\infty} \cos(tx)e^{-t} \, dt
\]

\[
= \frac{1}{\pi} \frac{1}{1 + x^2}
\]

by the result in equation (I). Hence \( X \sim \text{Cauchy} \).

**Diagnosing Discrete or Continuous Distributions**

(I) If

\[
\lim_{|t| \to \infty} \sup |\varphi_X(t)| = 1
\]

then \( X \) is often a discrete random variable. Technically, \( X \) may also have a singular distribution: see, or example

www.math.mcgill.ca/dstephens/556/Papers/Koopmans.pdf

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but such distributions are rarely encountered in practice.

(II) If
\[
\limsup_{|t| \to \infty} |\varphi_X(t)| = 0
\]
then \(X\) is continuous; consequently, if
\[
\lim_{|t| \to \infty} |\varphi_X(t)| = 0
\]
then \(X\) is continuous.

INTERPRETING THE CHARACTERISTIC FUNCTION.
To get a further understanding of characteristic function, we consider the inversion formulae. For discrete random variables defined on the integers, we have
\[
f_X(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ixt} \varphi_X(t) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(\pi t) - i \sin(\pi t)] \varphi_X(t) \, dt
\]
One way to think about this integral is via a discrete approximation; fix
\[
t_j = -\pi + \frac{2\pi j}{N} \quad j = 0, 1, \ldots, N
\]
and write
\[
f_X(x) \approx \frac{1}{2\pi} \left\{ \sum_{j=0}^{N} \cos(xt_j)\varphi_X(t_j) - i \sum_{j=0}^{N} \sin(xt_j)\varphi_X(t_j) \right\}
\]

(I) Suppose \(f_X\) is degenerate at \(x_0\), that is,
\[
f_X(x) = \begin{cases} 
1 & x = x_0 \\
0 & x \neq x_0
\end{cases}
\]
Then by elementary calculations
\[
\varphi_X(t) = \cos(x_0t) + i \sin(x_0t)
\]
so that
\[
\text{Re}(\varphi_X(t)) = \cos(x_0t) \quad \text{Im}(\varphi_X(t)) = \sin(x_0t)
\]
that is, pure sinusoids with period \(2\pi/x_0\).

(II) Suppose \(f_X\) is discrete, then as above
\[
\varphi_X(t) = \sum_{j=1}^{\infty} \cos(tx_j)f_X(x_j) + i \sum_{j=1}^{\infty} \sin(tx_j)f_X(x_j)
\]
so that
\[
\text{Re}(\varphi_X(t)) = \sum_{j=1}^{\infty} \cos(tx_j)f_X(x_j) \quad \text{Im}(\varphi_X(t)) = \sum_{j=1}^{\infty} \sin(tx_j)f_X(x_j)
\]
that is, a weighted sum of pure sinusoids with period \(2\pi/x_1, 2\pi/x_2, \ldots\), with weights determined by \(f_X\).