MATH 556: MATHEMATICAL STATISTICS I

ASSESSING INDEPENDENCE

Random variables $X_1, \ldots, X_d$ are independent if, **for all** $(x_1, \ldots, x_d)$

$$F_{X_1,\ldots,X_d}(x_1, \ldots, x_d) = \prod_{j=1}^{d} F_{X_j}(x_j)$$

or equivalently

$$f_{X_1,\ldots,X_d}(x_1, \ldots, x_d) = \prod_{j=1}^{d} f_{X_j}(x_j).$$

This definition is equivalent to saying that

$$f_{X_1|X_2,\ldots,X_d}(x_1|x_2 \ldots, x_d) = f_{X_1}(x_1)$$

for all possible selections of $x_1, \ldots, x_d$; note that the labelling of the variables is arbitrary, so this definition applies equivalently for any permutation of the labels.

The requirement that the factorizations hold **for all** $(x_1, \ldots, x_d)$ is important, and means that we typically need to consider the ranges of the random variables concerned. The following example illustrates this.

Example: Suppose, for $d = 2$, that X_1 and X_2 have joint pdf given by

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\pi} \qquad (x_1, x_2) \in D$$

where D is the unit disk, that is, the interior of the unit circle centered at (0,0). The joint pdf is constant on D, and the area of the disk is π, so the joint pdf integrates to 1. Marginally, we have that

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)\, dx_2 = \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \frac{1}{\pi}\, dx_2 = \frac{2}{\pi}\sqrt{1 - x_1^2} \quad 0 < x_1 < 1$$

and zero otherwise. Similarly

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)\, dx_1 = \int_{-\sqrt{1-x_2^2}}^{\sqrt{1-x_2^2}} \frac{1}{\pi}\, dx_1 = \frac{2}{\pi}\sqrt{1 - x_2^2} \quad 0 < x_2 < 1$$

and zero otherwise. Thus, clearly

$$f_{X_1,X_2}(x_1, x_2) \neq f_{X_1}(x_1) f_{X_2}(x_2)$$

and X_1 and X_2 are not independent.

However, it is also clear from inspection of the joint pdf that there are regions of $\mathbb{R}^2$ where the joint pdf is zero but where each marginal is non-zero. Considering X_1 and X_2 separately, it is clear that each variable can take values with positive probability anywhere on the interval $(-1, 1)$. However, under this model

$$P_X[X_1^2 + X_2^2 > 1] = 0$$

Example: Suppose, for $d = 2$, that X_1 and X_2 have joint pdf given by

$$f_{X_1,X_2}(x_1, x_2) = cx_1x_2 \quad (x_1, x_2) \in \mathbb{X}$$

and zero otherwise, where $\mathbb{X}$ is the set

$$\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < x_2 < 1\}.$$

We have that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)\; dx_2 dx_1 = 1 \qquad \Longrightarrow \qquad \int_0^1\int_{x_1}^1 cx_1x_2\; dx_2dx_1 = 1$$

as the joint pdf is zero outside of $\mathbb{X}$. Thus

$$c^{-1} = \int_0^1\int_{x_1}^1 x_1x_2\; dx_2dx_1 = \frac{1}{2}\int_0^1 x_1(1-x_1^2)\; dx_1 = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{1}{8}$$

so $c = 8$. It is tempting to observe that as $f_{X_1,X_2}(x_1, x_2)$ factorizes into a function of x_1 and a function of x_2, the two random variables are independent. However, from direct calculation, for $0 < x_1 < 1$

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)\; dx_2 = \int_{x_1}^1 x_1x_2\; dx_2$$

as the joint pdf is non-zero only when $x_1 < x_2$. Hence for $0 < x_1 < 1$

$$\begin{aligned} f_{X_1}(x_1) &= 8x_1\int_{x_1}^1 x_2\; dx_2 \\ &= 8x_1\left[\frac{1}{2}x_2^2\right]_{x_1}^1 \\ &= 4x_1(1-x_1^2) \end{aligned}$$

with the pdf zero otherwise. Similarly for $0 < x_2 < 1$

$$\begin{aligned} f_{X_2}(x_2) &= 8x_2\int_0^{x_2} x_1\; dx_1 \\ &= 8x_2\left[\frac{1}{2}x_1^2\right]_0^{x_2} \\ &= 4x_2^3 \end{aligned}$$

with the pdf zero otherwise. Clearly X_1 and X_2 are not independent. More precise definition of the joint pdf resolves the potential confusion: if we write

$$f_{X_1,X_2}(x_1, x_2) = 8\mathbb{1}_{\mathbb{X}}(x_1, x_2)x_1x_2 \quad (x_1, x_2) \in \mathbb{R}^2$$

using the indicator function, it is evident that this function does **not** factorize into a function of x_1 and a function of x_2 as the indicator contains both, and $\mathbb{X}$ is not a Cartesian product – we do not have

$$\mathbb{1}_{\mathbb{X}}(x_1, x_2) = \mathbb{1}_{(0,1)}(x_1) \times \mathbb{1}_{(0,1)}(x_2)$$

for example.