MATH 556 – EXAMPLE MID-TERM EXAMINATION

SOLUTIONS

1. We check the three requirements

   (i) Limit behaviour:
   $$\lim_{x \longrightarrow -\infty} F(x) = 0 \qquad \lim_{x \longrightarrow \infty} F(x) = 1.$$

   (ii) Non-decreasing property: if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.

   (iii) Right-continuity:
   $$\lim_{h \longrightarrow 0^+} F(x+h) = F(x).$$

   (a) Not a cdf: the function is not right-continuous at zero. 3 MARKS

   (b) Not a cdf: the function is decreasing in x for $x > 0$. 3 MARKS

   (c) This is the cdf for a continuous rv with support $\mathbb{R}$; we have
   $$\lim_{x \longrightarrow -\infty} F(x) = 0 \qquad \lim_{x \longrightarrow \infty} F(x) = 1$$
   and as the exponential function is continuous, $F(x)$ is continuous. It is also differentiable everywhere on $\mathbb{R}$, with derivative
   $$\lambda \frac{\exp\{\lambda(x-2)\}}{(1+\exp\{\lambda(x-2)\})^2} > 0$$
   so $F(x)$ is increasing on $\mathbb{R}$. 3 MARKS

   (d) Not a cdf: the function is not non-decreasing in x (as $F(x) = 0$ between the non-negative integers). 3 MARKS

   (e) This is a cdf if we define $F(0) = 1/2$ (this was omitted in error). Clearly we have
   $$\lim_{x \longrightarrow 0^-} F(x) = \lim_{x \longrightarrow 0^+} F(x) = \frac{1}{2}$$
   and
   $$\lim_{x \longrightarrow 2^-} F(x) = \lim_{x \longrightarrow 2^+} F(x) = 1$$
   so in fact this is the cdf of a continuous random variable. 3 MARKS

2. (a) From first principles, we have that Y is discrete with support $\mathbb{Y} = \{0, 1, 2, \ldots\}$, and for $y \in \mathbb{Y}$, we have

$$f_Y(y) = P_Y[Y = y] = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx = \int_0^{\infty} f_{Y|X}(y|x) f_X(x)\, dx$$

$$= \int_0^{\infty} e^{-x} \frac{x^y}{y!} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}\, dx$$

$$= \frac{1}{y!} \frac{1}{\Gamma(\alpha)} \int_0^{\infty} x^{y+\alpha-1} e^{-2x}\, dx$$

$$= \frac{1}{y!} \frac{1}{\Gamma(\alpha)} \frac{\Gamma(y+\alpha)}{2^{y+\alpha}}$$

as the integrand is proportional to a $Gamma(y + \alpha, 2)$ pdf. Thus

$$P_Y[Y = 0] = \frac{1}{2^{\alpha}}.$$

6 MARKS

(b) By iterated expectation, using the Distribution Formula Sheet

$$\mathbb{E}_Y[Y] = \mathbb{E}_X\left[\mathbb{E}_{Y|X}[Y|X]\right] = \mathbb{E}_X[X] = \alpha.$$

3 MARKS

(c) As Z is binary, we have

$$\mathbb{E}_Z[Z] = \mathbb{E}_Y\left[\mathbb{1}_{\{0\}}(Y)\right] = P_Y[Y = 0] = \frac{1}{2^{\alpha}}.$$

6 MARKS

3. We have that

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{a} & 0 \le x < a \\ 1 & a \ge 1 \end{cases}.$$

(a) We have that $Y = -\log(X/a) = -\log U$ say where $U \sim Uniform(0, 1)$. Hence from first principles, for $y > 0$,

$$F_Y(y) = P_Y[Y \le y] = P_U[-\log U \le y] = P_U[U \ge e^{-y}] = 1 - P_U[U < e^{-y}] = 1 - e^{-y}$$

so therefore $Y \sim Exponential(1)$, and hence $\mathbb{E}_Y[Y] = 1$ from the Distribution Formula Sheet.

4 MARKS

(b) For $0 < p < 1$

$$Q_X(p) = ap \qquad Q_Y(p) = -\log(1 - p).$$

4 MARKS

(c) By symmetry of form, we must have

$$P_{X_1,X_2}\left[X_1 > X_2\right] = \frac{1}{2}.$$

To verify this

$$P_{X_1,X_2}\left[X_1 > X_2\right] = \int_0^a \int_0^{x_1} f_{X_1,X_2}(x_1,x_2)\, dx_2 dx_1 = \int_0^a \int_0^{x_1} \frac{1}{a^2}\, dx_2 dx_1 = \frac{1}{a^2}\int_0^a x_1 dx_1 = \frac{1}{2}.$$

4 MARKS

(d) The transformation that can achieve this is

$$g(x) = \Phi^{-1}(x/a)$$

where $\Phi(.)$ is the standard Normal cdf. To verify this

$$F_Z(z) = P_Z[Z \le z] = P_X\left[\Phi^{-1}(X/a) \le z\right] = \mathrm{P}_X[X \le a\Phi(z)] = \Phi(z)$$

as required. 3 MARKS

4. (a) We have by independence

$$\mathbb{E}_{Z_1,Z_2}[Z_1^6 Z_2^9] = \mathbb{E}_{Z_1}[Z_1^6]\mathbb{E}_{Z_2}[Z_2^9] = 0$$

as $\mathbb{E}_{Z_2}[Z_2^9] = 0$, as it is an odd moment of the standard Normal distribution. 5 MARKS

(b) By definition

$$Cov_{X_1,X_2}[X_1,X_2] = \mathbb{E}_{X_1,X_2}[X_1X_2] - \mathbb{E}_{X_1}[X_1]\mathbb{E}_{X_2}[X_2]$$

where

$$\mathbb{E}_{X_1,X_2}[X_1X_2] = \mathbb{E}_{Z_1}[Z_1^3] = 0 \qquad \mathbb{E}_{X_1}[X_1] = 0$$

so therefore $Cov_{X_1,X_2}[X_1,X_2] = 0$. 4 MARKS

(c) In this case,

$$\begin{aligned}\log\frac{f_0(x)}{f_1(x)} &= \log\left[e^{-(x-\theta_0)^2/2}/e^{-(x-\theta_1)^2/2}\right]\\ &= \frac{1}{2}\left[(x-\theta_1)^2 - (x-\theta_0)^2\right]\\ &= \frac{1}{2}\left[2x(\theta_0-\theta_1) + \theta_1^2 - \theta_0^2\right].\end{aligned}$$

Thus as $\mathbb{E}_{f_0}[X] = \theta_0$ we have

$$KL(f_0,f_1) = \mathbb{E}_{f_0}\left[\log\frac{f_0(X)}{f_1(X)}\right] = \frac{1}{2}\left[2\theta_0(\theta_0-\theta_1) + \theta_1^2 - \theta_0^2\right] = \frac{1}{2}(\theta_0-\theta_1)^2.$$

Note that here $KL(f_0,f_1) = KL(f_1,f_0)$ which is not true in general.

6 MARKS