MATH 556 - EXERCISES 6: SOLUTIONS

1. (a) By direct calculation the mgf of $Y_i = X_i^2$ is

$$M_{Y_i}(t) = \mathbb{E}_{X_i}[e^{tX_i^2}] = \int_{-\infty}^{\infty} e^{tx^2} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}(x-\mu_i)^2\right\} dx$$

whenever $-1/2 < t < 1/2$, after completing the square in x in the exponent as

$$tx^2 - \frac{1}{2}(x-\mu_i)^2 = \left(t - \frac{1}{2}\right)(x - \mu_i/(1-2t))^2 - \frac{t/2}{(t-1/2)}\mu_i^2$$

and integrating the result, in which the integrand is proportional to a normal pdf, that is

$$M_{Y_i}(t) = \exp\left\{\frac{\mu_i^2 t}{(1-2t)}\right\} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{(1-2t)}{2}(x - \frac{\mu_i}{1-2t})^2\right\} dx$$

$$= \left(\frac{1}{1-2t}\right)^{1/2} \exp\left\{\frac{\mu_i^2 t}{(1-2t)}\right\}.$$

Hence, using the result for independent rvs, writing $\theta = \sum_{i=1}^{r} \mu_i^2$

$$M_Y(t) = \prod_{i=1}^{r} M_{Y_i}(t) = \left(\frac{1}{1-2t}\right)^{r/2} \exp\left\{\frac{\theta t}{1-2t}\right\}.$$

*The distribution of Y here is the **non-central Chisquared distribution** with r degrees of freedom and non-centrality parameter μ.*

(b) Many possible routes to compute the result. Could differentiate the mgf, or use direct calculation, or differentiate the cumulant generating function three times and evaluate at zero;

$$K_Y(t) = \log M_Y(t) = -\frac{r}{2}\log(1-2t) + \frac{\theta t}{1-2t}$$

so

$$K_Y^{(1)}(t) = \frac{r}{1-2t} + \frac{(1-2t)\theta + 2\theta t}{(1-2t)^2} = \frac{r}{1-2t} + \frac{\theta}{(1-2t)^2}$$

so that $\mu = \mathbb{E}_Y[Y] = K_Y^{(1)}(0) = r + \theta$.

$$K_Y^{(2)}(t) = \frac{2r}{(1-2t)^2} + \frac{4\theta}{(1-2t)^3}$$

so that $\sigma^2 = \text{Var}_{f_Y}[Y] = K_Y^{(2)}(0) = 2r + 4\theta = 2(r + 2\theta)$. Finally,

$$K_Y^{(3)}(t) = \frac{8r}{(1-2t)^3} + \frac{24\theta}{(1-2t)^4}$$

so that

$$\mathbb{E}_Y[(Y-\mu)^3] = K_Y^{(3)}(0) = 8r + 24\theta$$

yielding that

$$\varsigma = \frac{\mathbb{E}_Y[(Y-\mu)^3]}{\sigma^3} = \frac{8r + 24\theta}{(2r + 4\theta)^{3/2}} = \frac{2^{3/2}(r + 3\theta)}{(r + 2\theta)^{3/2}}$$

It is easy to verify that $K_X^{(3)}(0) = \mathbb{E}_X[(X-\mu)^3]$ by direct evaluation, complementing the results that $K_X^{(1)}(0) = \mathbb{E}_X[X]$ and $K_X^{(2)}(0) = \mathbb{E}_X[(X-\mu)^2]$.

2. (a) By iterated expectation, using the formula sheet to quote expectations for Gamma and Poisson

$$\mathbb{E}_X[X] = \mathbb{E}_N[\mathbb{E}_{X|N}[X|N]] = \mathbb{E}_N\left[\frac{N + r/2}{1/2}\right] = \frac{\mathbb{E}_N[N] + r/2}{1/2} = \frac{\lambda + r/2}{1/2} = 2\lambda + r$$

(b) By the same method of iterated expectation, for $-1/2 < t < 1/2$,

$$\begin{aligned} M_X(t) = \mathbb{E}_X[e^{tX}] &= \mathbb{E}_N[\mathbb{E}_{X|N}[e^{tX}|N]] = \mathbb{E}_N\left[\left(\frac{1/2}{1/2 - t}\right)^{N+r/2}\right] \\ &= \left(\frac{1/2}{1/2 - t}\right)^{r/2} \mathbb{E}_N\left[\left(\frac{1/2}{1/2 - t}\right)^{N}\right] \\ &= \left(\frac{1}{1 - 2t}\right)^{r/2} G_N\left(\frac{1}{1 - 2t}\right) \\ &= \left(\frac{1}{1 - 2t}\right)^{r/2} \exp\left\{\lambda\left(\frac{1}{1 - 2t} - 1\right)\right) = \left(\frac{1}{1 - 2t}\right)^{r/2} \exp\left\{\frac{2\lambda t}{1 - 2t}\right\} \end{aligned}$$

*The distribution of Y here is again the **non-central Chisquared distribution** with r degrees of freedom and non-centrality parameter λ, identical to the form found in Q1 (a).*

3. By iterated expectation

$$\mathbb{E}_{X_1}[X_1] = \mathbb{E}_M\left[\mathbb{E}_{X_1|M}[X_1|M]\right] = \mathbb{E}_M[M] = \mu$$

and

$$\mathbb{E}_{X_1}[X_1^2] = \mathbb{E}_M\left[\mathbb{E}_{X_1|M}[X_1^2|M]\right] = \mathbb{E}_M\left[M^2 + \sigma^2\right] = \mu^2 + \tau^2 + \sigma^2$$

so that

$$\text{Var}_{X_1}[X_1] = \mathbb{E}_{X_1}[X_1^2] - \{\mathbb{E}_{X_1}[X_1]\}^2 = \tau^2 + \sigma^2.$$

By symmetry of form, $\mathbb{E}_{X_2}[X_2] = \mu$ and $\text{Var}_{X_2}[X_2] = \tau^2 + \sigma^2$. Now,

$$\mathbb{E}_{X_1,X_2}[X_1X_2] = \mathbb{E}_M\left[\mathbb{E}_{X_1,X_2|M}[X_1X_2|M]\right] = \mathbb{E}_M\left[\mathbb{E}_{X_1|M}[X_1|M] \times \mathbb{E}_{X_2|M}[X_2|M]\right]$$

by conditional independence. Therefore

$$\mathbb{E}_{X_1,X_2}[X_1X_2] = \mathbb{E}_M[M \times M] = \mathbb{E}_M\left[M^2\right] = \mu^2 + \tau^2$$

Hence

$$\text{Cov}_{X_1,X_2}[X_1, X_2] = \mathbb{E}_{X_1,X_2}[X_1X_2] - \mathbb{E}_{X_1}[X_1]\mathbb{E}_{X_2}[X_2] = \mu^2 + \tau^2 - \mu^2 = \tau^2$$

and

$$\text{Corr}_{X_1,X_2}[X_1, X_2] = \frac{\text{Cov}_{X_1,X_2}[X_1, X_2]}{\sqrt{\text{Var}_{X_1}[X_1]\text{Var}_{X_2}[X_2]}} = \frac{\tau^2}{\tau^2 + \sigma^2}$$

X_1 and X_2 are not independent; their covariance is non zero.