MATH 556 - EXERCISES 5: SOLUTIONS

1. (a) This is not an Exponential Family distribution; the support is parameter dependent.

   (b) This is an EF distribution with $m = 1$:

$$f(x;\theta) = \frac{\mathbb{1}_{\{1,2,3,\ldots\}}(x)}{x} \frac{-1}{\log(1-\theta)} \exp\{x \log\theta\} = \exp\{c(\theta)T(x) - A(\theta)\}h(x)$$

   - $h(x) = \dfrac{\mathbb{1}_{\{1,2,3,\ldots\}}(x)}{x}$
   - $A(\theta) = \log(-\log(1-\theta))$
   - $c(\theta) = \log(\theta)$
   - $T(x) = x$

   so the natural parameter is $\eta = \log(\theta)$.

2. (a) Suppose that $\eta_1, \eta_2 \in \mathcal{H}$ and $0 \leq t \leq 1$. Then

$$\begin{aligned}
\int h(x) e^{(t\eta_1 + (1-t)\eta_2)^\top T(x)}\, dx &= \int h(x) e^{(t\eta_1)^\top T(x)} e^{((1-t)\eta_2)^\top T(x)}\, dx \\
&\leq \left\{\int h(x) e^{(t\eta_1)^\top T(x)} dx\right\}\left\{\int h(x) e^{((1-t)\eta_2)^\top T(x)}\, dx\right\} \\
&\leq \left\{\int h(x) e^{\eta_1^\top T(x)} dx\right\}^t \left\{\int h(x) e^{\eta_2^\top T(x)}\, dx\right\}^{(1-t)} < \infty
\end{aligned}$$

   so $t\eta_1 + (1-t)\eta_2 \in \mathcal{H}$.

   (b) By inspection

$$\log \frac{f_X(x;\eta_1)}{f_X(x;\eta_2)} = (\eta_1 - \eta_2)T(x) - (K(\eta_1) - K(\eta_2))$$

   Note that this ratio is zero for all x **if and only if** $\eta_1 = \eta_2$, unless $T(x)$ is a constant, t_0, say, for all x. In this latter case, we have that

$$K(\eta) = \log\left\{\int h(x) \exp\{\eta t_0\}\, dx\right\} = \eta t_0$$

   in which case

$$\log \frac{f_X(x;\eta_1)}{f_X(x;\eta_2)} = (\eta_1 - \eta_2)t_0 - (\eta_1 t_0 - \eta_2 t_0) = 0$$

   also, for any η_1 and η_2. Hence we can conclude that the EF model is *identifiable*

$$f_X(x;\eta_1) = f_X(x;\eta_2) \qquad \Longleftrightarrow \qquad \eta_1 = \eta_2$$

   unless $T(X)$ has a degenerate distribution (for a value $\eta_0 \in \mathcal{H}$).

3. We have

$$f_X(x;\psi,\gamma) = \mathbb{1}_{(0,\infty)}(x) \sqrt{\frac{1}{2\pi\gamma x^3}} \exp\left\{-\frac{1}{2}\psi^2\gamma x + \psi - \frac{1}{2\gamma x}\right\}$$

   for $\psi, \gamma > 0$ and

(a) This is NOT a location-scale family. For the family to be a location-scale family, we must be able to make a transform of the form

$$Z = \frac{X - \mu}{\sigma}$$

with the result that the distribution of Z does not depend on any parameters. The presence of the $1/x$ term renders the required linear transformation impossible.

(b) This IS an Exponential Family distribution; we may write the transparent parameterization

$$f_X(x;\psi,\gamma) = h(x)\exp\left\{(c_1(\theta_1), c_2(\theta_2))\begin{pmatrix}T_1(x)\\T_2(x)\end{pmatrix} - A(\theta)\right\}$$

where

- $h(x) = \mathbb{1}_{(0,\infty)}(x)x^{-3/2}(2\pi)^{-1/2}$
- $T_1(x) = x$, $T_2(x) = 1/x$.
- $c_1(\theta) = -\dfrac{1}{2}\psi^2\gamma$ and $c_2(\theta) = -\dfrac{1}{2\gamma}$.
- $A(\theta) = -\psi + \dfrac{1}{2}\log\gamma$.

(c) Using the score result,we see that

$$\mathbb{E}_X\left[\frac{\partial c_1(\theta)}{\partial\psi}X + \frac{\partial c_2(\theta)}{\partial\psi}\frac{1}{X}\right] = \frac{\partial A(\theta)}{\partial\psi}$$

and

$$\mathbb{E}_X\left[\frac{\partial c_1(\theta)}{\partial\gamma}X + \frac{\partial c_2(\theta)}{\partial\gamma}\frac{1}{X}\right] = \frac{\partial A(\theta)}{\partial\gamma}$$

or equivalently

$$\mathbb{E}_X\left[-\psi\gamma X + 0\frac{1}{X}\right] = -1 \qquad \therefore \qquad \mathbb{E}_X[X] = \frac{1}{\psi\gamma}$$

and

$$\mathbb{E}_X\left[-\frac{1}{2}\psi^2 X + \frac{1}{2\gamma^2}\frac{1}{X}\right] = \frac{1}{2\gamma} \qquad \therefore \qquad \mathbb{E}_X\left[\frac{1}{X}\right] = \gamma + \psi\gamma$$

Note that we may further rewrite the density

$$f_X(x;\phi_1,\phi_2) = \mathbb{1}_{(0,\infty)}(x)\sqrt{\frac{\phi_1}{2\pi x^3}}\exp\left\{-\frac{\phi_1}{2}\frac{(x-\phi_2)^2}{\phi_2^2 x}\right\}$$

where

$$\phi_1 = \frac{1}{\gamma} \qquad \phi_2 = \frac{1}{\psi\gamma}$$

rendering

$$\mathbb{E}_X[X] = \phi_2 \qquad \mathbb{E}_X\left[\frac{1}{X}\right] = \frac{1}{\phi_1} + \frac{1}{\phi_2}$$