MATH 556 - EXERCISES 1– SOLUTIONS

1. (a) The density must integrate to 1. By elementary calculations

$$\int_{-\infty}^{\infty} \exp\{-\lambda|x-\theta|\}\, dx = \int_{-\infty}^{\infty} \exp\{-\lambda|z|\}\, dz = 2\int_{0}^{\infty} \exp\{-\lambda z\}\, dz$$

first by making the change of variables from x to $z = x-\theta$, and then noting that the resulting integrand is an integrable even function around zero. Thus the integral equals $2/\lambda$, so we must have $c = \lambda/2$

(b) The cdf takes slightly different forms either side of θ. First, note that the density is symmetric about θ so we have

$$F_X(x) = \begin{cases} \dfrac{1}{2} - \dfrac{1}{2}(1 - \exp\{\lambda(x-\theta)\} & x \le \theta \\ \dfrac{1}{2} + \dfrac{1}{2}(1 - \exp\{-\lambda(x-\theta)\} & x > \theta \end{cases}$$

(c) The quantile function takes slightly different forms either side of $p = 1/2$. Again by symmetry, we have

$$Q_X(p) = \begin{cases} \theta + \dfrac{1}{\lambda}\log(2p) & p \le 1/2 \\ \theta - \dfrac{1}{\lambda}\log(2(1-p)) & p > 1/2 \end{cases}$$

(d) By symmetry, and the fact that the expectation is finite, we conclude that $\mu = \mathbb{E}_X[X] = \theta$;

(e) Changing variables in the integral from x to $z = x - \theta$, we have

$$\text{Var}_X[X] = \int_{-\infty}^{\infty} (x-\theta)^2\, \frac{\lambda}{2} \exp\{-\lambda|x-\theta|\} dx = \int_{-\infty}^{\infty} z^2\, \frac{\lambda}{2} \exp\{-\lambda|z|\} dz$$

and so the variance of X is equal to the variance of $Z = X - \theta$, so using

$$f_Z(z) = \frac{\lambda}{2} \exp\{-\lambda|z|\} \qquad z \in \mathbb{R}$$

we have

$$\begin{aligned} \mathbb{E}_Z[Z^2] &= \frac{\lambda}{2}\int_{-\infty}^{0} z^2 \exp\{\lambda z\}\, \mathrm{d}z + \frac{\lambda}{2}\int_{0}^{\infty} z^2 \exp\{-\lambda z\}\, \mathrm{d}z \\ &= \lambda \int_{0}^{\infty} z^2 \exp\{-\lambda z\}\, \mathrm{d}z = \lambda\frac{\Gamma(3)}{\lambda^3} = \frac{2}{\lambda^2} \end{aligned}$$

by the fact that in the integral the integrand is proportional to a Gamma pdf.

2. Recall that the quantile function is defined by

$$Q_X(p) = \inf\{x : F_X(x) \ge p\} \quad 0 < p < 1$$

so to find $Q_X(p)$, we need to find the smallest x such that $F_X(x) \ge p$. If the cdf is (absolutely) continuous and strictly monotonic, we can obtain the quantile function simply by noting that

$$Q_X(p) = x \quad \Longleftrightarrow \quad F_X(x) = p.$$

(a) We have that

$$F_X(x) = 1 - \exp\{-\beta x^\alpha\} \qquad x > 0$$

with $F_X(0) = 0$ for $x \leq 0$. Therefore, this is strictly monotonic on $\mathbb{R}^+$, and by direct calculation

$$Q_X(p) = \left\{-\frac{1}{\beta}\log(1-p)\right\}^{1/\alpha} \qquad 0 < p < 1$$

(b) We deduce directly that $c = 1/10$, and hence that

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \dfrac{\lfloor \min\{x, 10\} \rfloor}{10} & x \geq 1 \end{cases}.$$

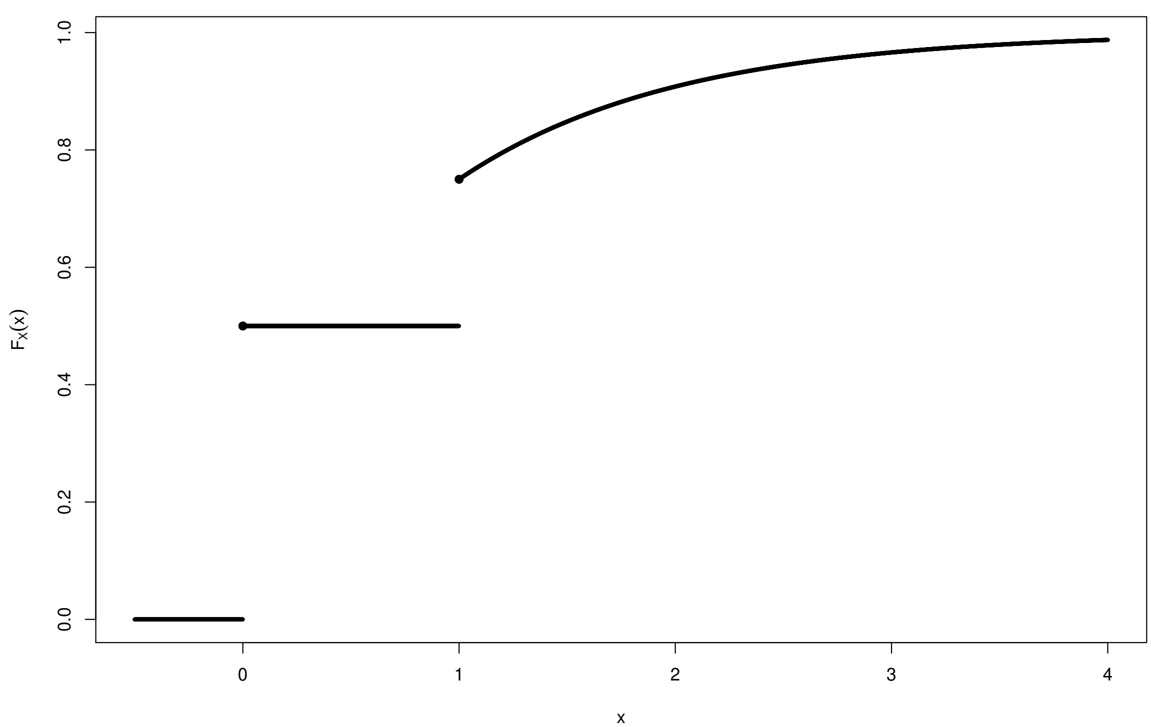
Hence

$$Q_X(p) = \lceil 10p \rceil \qquad 0 < p < 1.$$

(c) This is a 'mixed-type' distribution, but by right-continuity at $x = 1$ we must have

$$\frac{3}{4} = 1 - c \qquad \Longrightarrow \qquad c = \frac{1}{4}.$$

and the cdf is depicted in the following figure:



so therefore

$$Q_X(p) = \begin{cases} 0 & 0 < p \leq 0.5 \\ 1 & 0.5 < p \leq 0.75 \\ 1 - \log(4(1-p)) & 0.75 < p < 1 \end{cases}$$

3. (a) For $y > 0$

$$F_{Y_1}(y) = P_{Y_1}[Y_1 \leq y] = P_X[X^2 \leq y] = P_{Y_1}[-\sqrt{y} \leq X \leq \sqrt{y}] = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

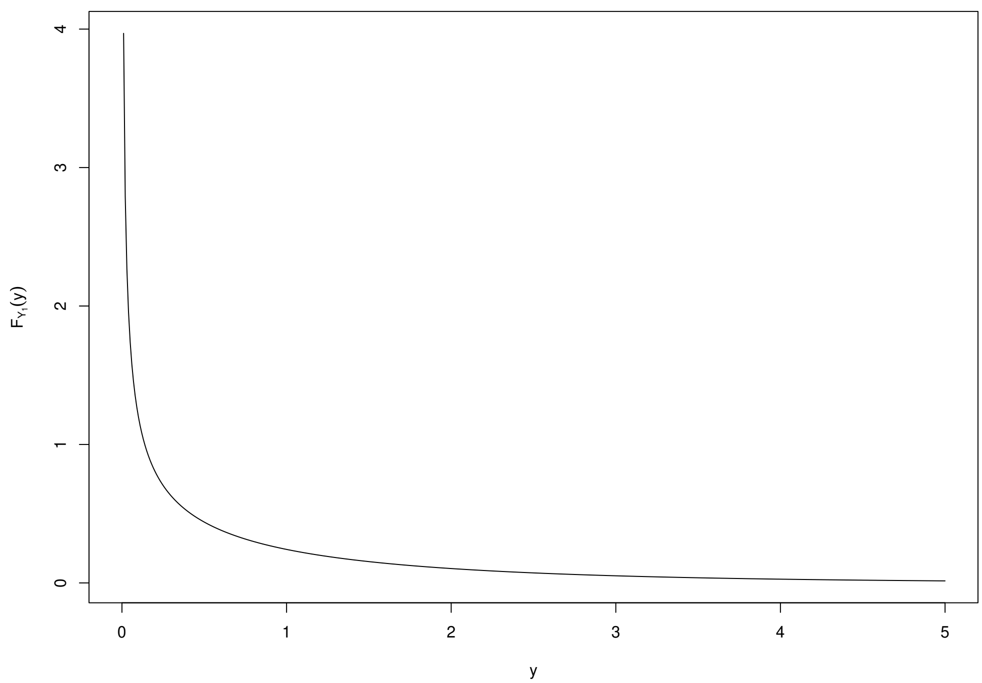
so therefore by differentiation, for $y > 0$

$$f_{Y_1}(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) = \frac{1}{\sqrt{y}} f_X(\sqrt{y})$$

as $f_X(.)$ is symmetric around zero. That is

$$f_{Y_1}(y) = \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{y}{2}\right\} \qquad y > 0$$

and zero otherwise.



(b) For $y > 0$

$$F_{Y_2}(y) = P_{Y_2}[Y_2 \leq y] = P_X[|X| \leq y] = P_{Y_2}[-y \leq X \leq y] = F_X(y) - F_X(-y)$$

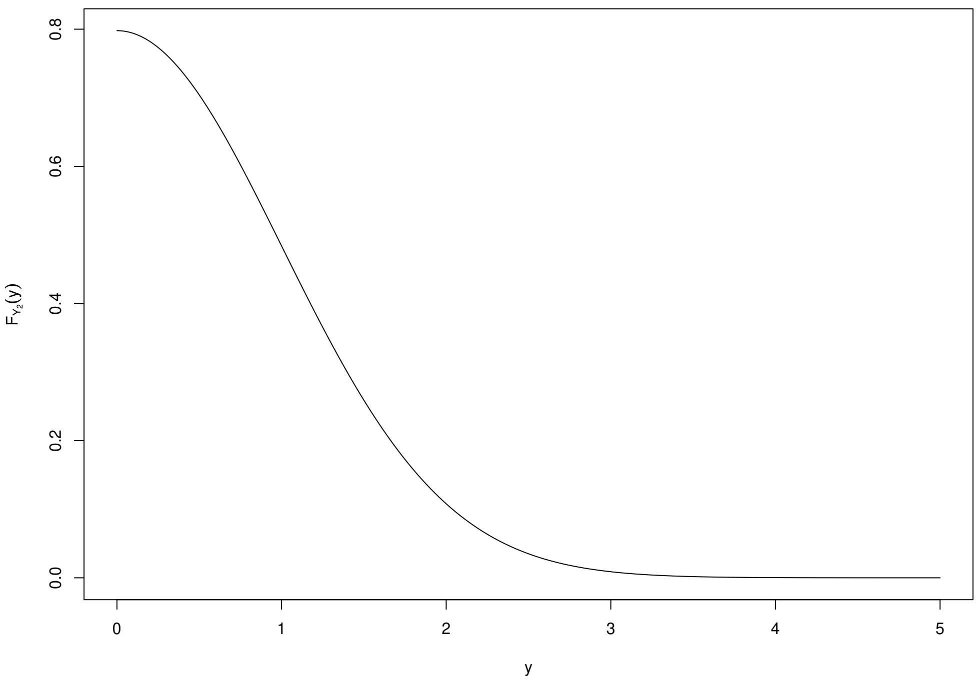
so therefore by differentiation, for $y > 0$

$$f_{Y_2}(y) = f_X(y) + f_X(-y) = 2f_X(y)$$

as $f_X(.)$ is symmetric around zero. That is

$$f_{Y_2}(y) = \frac{2}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \qquad y > 0$$

and zero otherwise.



(c) We have

$$\begin{aligned} F_{Y_3}(y) = P_{Y_3}[Y_3 \leq y] &= P_X[2X - X^2 \leq y] \\ &= P_X[X^2 - 2X + y \geq 0] \\ &= P_X[(X - a_1(y))(X - a_2(y)) \geq 0] \end{aligned}$$

say, where

$$(a_1(y), a_2(y)) = \frac{2 \pm \sqrt{4(1-y)}}{2} = 1 \pm \sqrt{1-y}$$

provided $y \leq 1$; if $y > 1$, $P_X[X^2 - 2X + y \geq 0] = 1$. Thus for $y < 1$,

$$F_{Y_3}(y) = P_X[X \leq a_1(y)] + P_X[X \geq a_2(y)] = F_X(a_1(y)) + 1 - F_X(a_2(y))$$

and hence

$$f_{Y_3}(y) = \frac{1}{2\sqrt{1-y}} f_X(1 - \sqrt{1-y}) + \frac{1}{2\sqrt{1-y}} f_X(1 + \sqrt{1-y})$$

(d) The function $F_X(.)$ maps onto $(0, 1)$, so for $0 < y < 1$

$$F_{Y_4}(y) = P_{Y_4}[Y_4 \leq y] = P_X[F_X(X) \leq y] = P_X[X \leq F_X^{-1}(y)] = F_X(F_X^{-1}(y)) = y$$

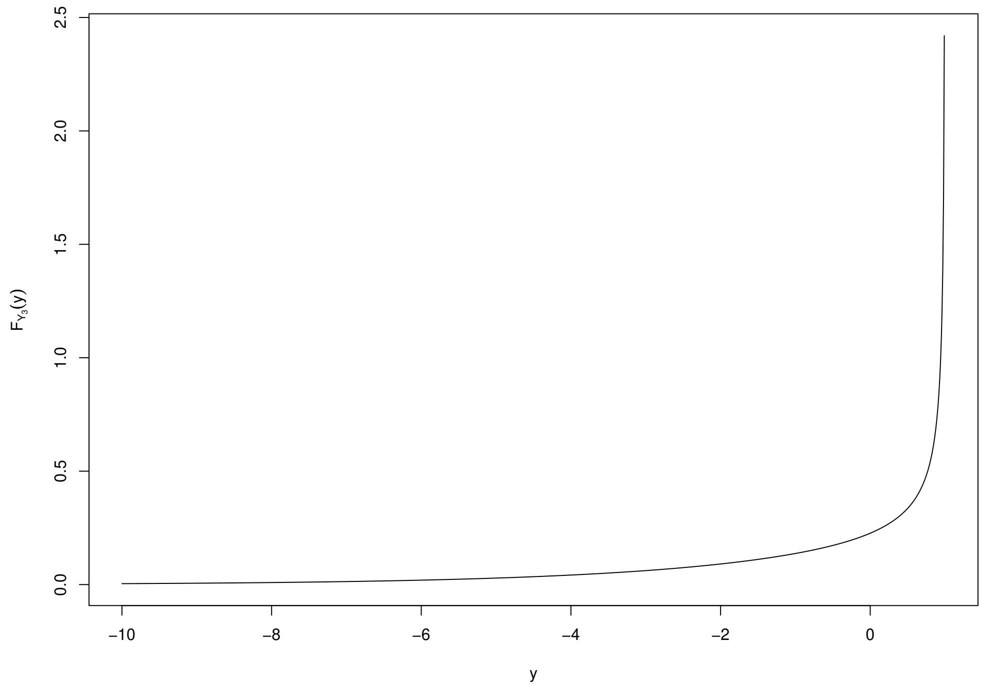
so therefore

$$f_{Y_4}(y) = 1 \qquad 0 < y < 1$$

and zero otherwise.

4. The cdf of the $Pareto(\theta, \alpha)$ distribution is

$$F_X(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha} \qquad x > 0$$



with $F_X(x) = 0$ for $x \leq 0$. This is strictly increasing on $\mathbb{R}^+$, and so the quantile function can be computed as

$$Q_X(p) = \theta(\{(1-p)\}^{-1/\alpha} - 1)$$

Now, recall question 3. (d); this says result implies that if X has strictly increasing cdf F_X, then the transformed variable $U = F_X(X)$ has a distribution that is uniform on (0,1) – we write $U \sim Uniform(0,1)$. But if F_X is strictly increasing, then the inverse function F_X^{-1} is well-defined and corresponds precisely to the quantile function. Consequently, we must have that if $U \sim Uniform(0,1)$, then the transformed random variable $X = F_X^{-1}(U)$ has cdf F_X.

Therefore, consider setting

$$X = \theta(\{(1-(1-\exp\{-Z\}))\}^{-1/\alpha} - 1)$$

or

$$X = \theta(\exp\{Z/\alpha\} - 1)$$

as we require

$$\mathrm{P}_Z[g(Z) \leq x] = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}$$

for $x > 0$, but

$$\mathrm{P}_Z[g(Z) \leq x] \equiv \mathrm{P}_Z[Z \leq g^{-1}(x)] = 1 - \exp\{-g^{-1}(x)\}$$

dictates that

$$\exp\{-g^{-1}(x)\} = \left(\frac{\theta}{\theta + x}\right)^{\alpha}$$

or

$$g^{-1}(x) = -\alpha \log\theta + \alpha\log(\theta + x)$$

which yields the solution.

5. We have

$$\mathbb{E}_Y[Y] \equiv \mathbb{E}_X[\{F_X(X)\}^k] = \int_{-\infty}^{\infty} \{F_X(x)\}^k f_X(x)\, dx = \left[\frac{1}{k+1}\{F_X(x)\}^{k+1}\right]_{-\infty}^{\infty} = \frac{1}{k+1}.$$

where the penultimate step follows as $f_X(x) = dF_X(x)/dx$, and the final step follows by properties of the cdf that $F_X(-\infty) = 0$ and $F_X(\infty) = 1$.

6. For joint density defined on the unit cube $(0,1)^3$.

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = c(1 - \sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3))$$

and zero otherwise, for some constant c.

(a) We have for $0 < x_1, x_2 < 1$

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \int_0^1 c(1 - \sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3))\ \mathrm{d}x_3 \\ &= c - c\sin(2\pi x_1)\sin(2\pi x_2)\int_0^1 \sin(2\pi x_3)\ \mathrm{d}x_3 \\ &= c \end{aligned}$$

with the joint pdf for (X_1, X_2) zero elsewhere. The joint pdf must integrate to 1, and thus $c = 1$, and by direct calculation

$$f_{X_1}(x_1) = 1 \qquad 0 < x_1 < 1$$

with the same result for X_2, so X_1 and X_2 are marginally uniform. As

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2) = 1 \qquad (x_1,x_2) \in (0,1)\times(0,1)$$

and

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2) = 0 \qquad (x_1,x_2) \notin (0,1)\times(0,1)$$

X_1 and X_2 are independent.

(b) (X_1, X_2, X_3) are **not** independent as the joint pdf does not factorize into the product of marginals, which is a necessary condition for independence. We can see this, as the function

$$(1 - \sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3))$$

does not factorize.