MATH 556 – FALL 2019
MID-TERM EXAMINATION

SOLUTIONS

1. (a) Not a pmf: function sums to 3. 3 MARKS

   (b) Valid pmf: support is finite, and function is bounded on that support, so c is finite. 3 MARKS

   (c) Valid pdf: function is integrable on $\mathbb{R}$ as $\Phi(x)$ is bounded, and function is positive on $\mathbb{R}$. 3 MARKS

   (d) Not a cdf: we have $\lim\limits_{x \longrightarrow \infty} F(x) = \frac{1}{2} < 1$. 3 MARKS

   (e) Valid cdf with $c_1 = c_2 = 1/\sqrt{2}$: function satisfies the three conditions (limit conditions, non-decreasing, right-continuous, even at $x = 1$, where $F(x) = 1/\sqrt{2}$). 3 MARKS

2. (a) We have that for $y \in \mathbb{R}$

$$f_Y(y) = \sum_{x=0}^{1} f_{Y|X}(y|x) f_X(x) = \phi(y+1)(1-\theta) + \phi(y-1)\theta$$

where $\phi(x)$ is the standard Normal pdf. 3 MARKS

For the expectation, by direct calculation

$$\mathbb{E}_Y[Y] = \int_{-\infty}^{\infty} y(\phi(y+1)(1-\theta) + \phi(y-1)\theta)\ dy$$

and by standard properties of integrals we have

$$\mathbb{E}_Y[Y] = (1-\theta)\int_{-\infty}^{\infty} y\phi(y+1)\ dy + \theta\int_{-\infty}^{\infty} y\phi(y-1)\ dy$$

which, by changing variables in the integrals, we can rewrite as

$$\mathbb{E}_Y[Y] = (1-\theta)\int_{-\infty}^{\infty} (z-1)\phi(z)\ dz + \theta\int_{-\infty}^{\infty} (z+1)\phi(z))\ dz.$$

Then by properties of the standard Normal, we have

$$\mathbb{E}_Y[Y] = ((1-\theta)\times(-1)) + (\theta \times 1) = 2\theta - 1.$$

Similarly

$$\begin{aligned}
\mathbb{E}_Y[Y^2] &= (1-\theta)\int_{-\infty}^{\infty} y^2\phi(y+1)\ dy + \theta\int_{-\infty}^{\infty} y^2\phi(y-1)\ dy \\
&= (1-\theta)\int_{-\infty}^{\infty} (z-1)^2\phi(z)\ dz + \theta\int_{-\infty}^{\infty} (z+1)^2\phi(z)\ dz \\
&= \mathbb{E}_Z[Z^2] + (2\theta-1)\mathbb{E}_Z[Z] + 1
\end{aligned}$$

where $Z \sim Normal(0,1)$. Thus $\mathbb{E}_Y[Y^2] = 2$, and thus

$$\text{Var}_Y[Y] = \mathbb{E}_Y[Y^2] - \{\mathbb{E}_Y[Y]\}^2 = 2 - (2\theta-1)^2 = 1 + 4\theta(1-\theta).$$

Note: we can also use the method of *iterated expectation* here. 6 MARKS

(b) Z can only take values on $\{0,1,2,3\}$ with positive probability. We have

$$f_Z(z) = \begin{cases} (1-\theta)^2 & z = 0 \\ \theta(1-\theta) & z = 1 \\ \theta(1-\theta) & z = 2 \\ \theta^2 & z = 3 \\ 0 & \text{otherwise} \end{cases}$$

6 MARKS

3. (a) (i) We have that $Q_X(p) = \inf\{x : F_X(x) \geq p\}$, so

$$Q_X(p) = \begin{cases} -1 & 0 < p \leq 1/4 \\ 0 & 1/4 < p \leq 3/4 \\ 1 & 3/4 < p < 1 \end{cases}$$

5 MARKS

(ii) Y can only take values on $\{0, 1\}$, and $Y \sim Bernoulli(1/2)$. 2 MARKS

(iii) The pmf is symmetric about zero, so $\mathbb{E}_X[X^3] = 0$. 3 MARKS

(b) We have

$$\log \frac{f_0(x)}{f_1(x)} = \log \left[\frac{\lambda_0^x e^{-\lambda_0}}{\lambda_1^x e^{-\lambda_1}}\right] = x \log(\lambda_0/\lambda_1) - (\lambda_0 - \lambda_1)$$

Now, it is evident that the KL divergence is the expectation of this logged quantity under the distribution f_0, and as $\mathbb{E}_{f_0}[X] = \lambda_0$ we have

$$KL(f_0, f_1) = \mathbb{E}_{f_0}\left[\log \frac{f_0(X)}{f_1(X)}\right] = \mathbb{E}_{f_0}\left[X \log(\lambda_0/\lambda_1) - (\lambda_0 - \lambda_1)\right] = \lambda_0 \log(\lambda_0/\lambda_1) - (\lambda_0 - \lambda_1).$$

5 MARKS

4. (a) We have

$$P_{X_1,X_2}\left[\frac{X_1}{X_2} > 1\right] = P_{X_1,X_2}\left[X_1 > X_2\right] = \frac{1}{2}.$$

as X_1 and X_2 are independent and identically distributed.

3 MARKS

(b) We have for fixed $y < 0$,

$$P_Y[Y \leq y] = P_{X_1,X_2}[X_1 - X_2 \leq y] = \int_0^\infty \int_{x_1-y}^\infty e^{-x_1} e^{-x_2}\, dx_2 dx_1$$

as for $y < 0$, the set A_y is the region in the positive quadrant above the line $x_1 - x_2 = y$. Thus

$$F_Y(y) = \int_0^\infty e^{-x_1}\left\{\int_{x_1-y}^\infty e^{-x_2}\, dx_2\right\} dx_1 = \int_0^\infty e^{-x_1} e^{-(x_1-y)} dx_1$$

$$= e^y \int_0^\infty e^{-2x_1} dx_1 = \frac{1}{2} e^y$$

For $y > 0$, note that

$$P_Y[Y \geq y] = P_Y[-Y \leq -y] = P_{X_1,X_2}[X_2 - X_1 \leq -y]$$

but by symmetry of form, $X_2 - X_1$ has the same distribution as $X_1 - X_2$. Thus for $y > 0$

$$F_Y(y) = 1 - P_Y[Y \geq y] = 1 - F_Y(-y) = 1 - \frac{1}{2} e^{-y}$$

and hence we have on differentiation (separately for $y > 0$ and $y < 0$) that

$$f_Y(y) = \frac{1}{2} e^{-|y|} \qquad y \in \mathbb{R}.$$

This is the *Double Exponential* (or *Laplace*) distribution. 12 MARKS