MATH 556 - EXERCISES 6

Not for Assessment.

1. Suppose that $X_1, \ldots, X_r$ are independent random variables such that, for each i, $X_i \sim N(\mu_i, 1)$, for fixed constants $\mu_1, \ldots, \mu_r$.

   (a) Find the mgf of random variable Y defined by

   $$Y = \sum_{i=1}^{r} X_i^2.$$

   (b) Find the skewness of Y, ς, where

   $$\varsigma = \frac{\mathbb{E}_Y[(Y-\mu)^3]}{\sigma^3}$$

   where μ and σ^2 are the expectation and variance of f_Y.

2. Consider the three-level hierarchical model:

   $$\begin{array}{lll} \text{LEVEL 3 :} & \lambda > 0, r \in \{1, 2, \ldots\} & \text{Fixed parameters} \\ \text{LEVEL 2 :} & N \sim Poisson(\lambda) & \\ \text{LEVEL 1 :} & X|N = n \sim Gamma(n + r/2, 1/2) & \end{array}$$

   Find

   (a) The expectation of X, $\mathbb{E}_X[X]$,
   (b) The mgf of X, $M_X(t)$.

3. As a generalization of the model considered in lectures, consider the three-level hierarchical model:

   $$\begin{array}{lll} \text{LEVEL 3 :} & \mu \in \mathbb{R}, \tau, \sigma > 0 & \text{Fixed parameters} \\ \text{LEVEL 2 :} & M \sim Normal(\mu, \tau^2) & \\ \text{LEVEL 1 :} & X_1, X_2|M = m \sim Normal(m, \sigma^2) & \end{array}$$

   where X_1 and X_2 are conditionally independent given M, denoted

   $$X_1 \perp X_2 \mid M.$$

   Using the law of iterated expectation, find the (marginal) covariance and correlation between X_1 and X_2. Are X_1 and X_2 (marginally) independent ? Justify your answer.