MATH 556 - ASSIGNMENT 4

To be handed in not later than 11.59pm, 5th December 2022.
Please submit your solutions as pdf via myCourses.

1. Consider the three-level hierarchical model

LEVEL 3 :	$\theta = (\theta_1, \theta_2) \in \mathbb{R}^+ \times \mathbb{R}^+$	Fixed
LEVEL 2 :	$X \sim Gamma(\theta_1, \theta_2)$	
LEVEL 1 :	$Y_1, \ldots, Y_n \vert X = x \sim Poisson(x)$	$Y_1, \ldots, Y_n$ independent given X

   (a) Find the (marginal) joint pmf of $Y_1, \ldots, Y_n$. 4 Marks

   (b) Find the marginal pmf of Y_1. 2 Marks

   (c) Find the correlation between Y_1 and Y_2. 4 Marks

2. For $n \geq 1$ random variables $X_1, \ldots, X_n$, the *order statistics*, $Y_1, \ldots, Y_n$, are defined by

$$Y_i = X_{(i)} - \text{"the } i\text{th smallest value in } X_1, \ldots, X_n\text{"}$$

   for $i = 1, \ldots, n$. For example

$$Y_1 = X_{(1)} = \min\{X_1, \ldots, X_n\} \qquad\qquad Y_n = X_{(n)} = \max\{X_1, \ldots, X_n\}.$$

   For $X_1, \ldots, X_n$ independently distributed from continuous distribution with pdf f_X, the joint pdf of order statistics $Y_1, \ldots, Y_n$ can be shown to be

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = n! f_X(y_1) \ldots f_X(y_n) \qquad y_1 < \ldots < y_n$$

   and zero otherwise.

   (a) Suppose X_1, X_2, X_3 are independent random variables having an $Exponential(1)$ distribution. Find the distribution of the second order statistic, Y_2, that is, the second smallest of X_1, X_2, X_3.

   5 Marks

   (b) Suppose $X_1, \ldots, X_n$ are independent continuous random variables with cdf F_X

$$F_X(x) = 1 - x^{-1} \qquad x \geq 1$$

   and zero otherwise.

   Show that $Z_n = \min\{X_1, \ldots, X_n\}$ has a **degenerate** distribution in the limit as $n \longrightarrow \infty$, that is, that

$$\lim_{n \longrightarrow \infty} P_{Z_n}[Z_n = c] = 1$$

   for some c to be identified, but that there exists a sequence of real values $\{\alpha_n\}$ such that $U_n = Z_n^{\alpha_n}$ has distribution F_X for each n.

   5 Marks

   Hint: for the first part, having identified c, show that

$$P_{Z_n}[Z_n < c] + P_{Z_n}[Z_n > c] \longrightarrow 0$$

   as $n \longrightarrow \infty$.