MATH 556 - ASSIGNMENT 3 – SOLUTIONS

1. *Suppose that Z_1 and Z_2 are independent random variables having a $Normal(0,1)$ distribution.*

   (a) *Find the joint pdf of random variables X_1 and X_2 defined by*

$$X_1 = \frac{Z_1}{Z_2} \qquad\qquad X_2 = Z_1 + Z_2.$$

The inverse transformation is

$$Z_1 = \frac{X_1 X_2}{1+X_1} \qquad Z_2 = \frac{X_2}{1+X_1}$$

and so the Jacobian is

$$\left| \det \begin{bmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_1}{\partial x_2} \\ \dfrac{\partial z_2}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \end{bmatrix} \right| = \left| \det \begin{bmatrix} \dfrac{x_2}{(1+x_1)^2} & \dfrac{x_1}{1+x_1} \\ -\dfrac{x_2}{(1+x_1)^2} & \dfrac{1}{(1+x_1)^2} \end{bmatrix} \right| = \frac{|x_2|}{(1+x_1)^2}$$

and hence, by the independence of Z_1 and Z_2 the joint pdf is

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= f_{Z_1}\left(\frac{x_1x_2}{1+x_1}\right) f_{Z_2}\left(\frac{x_2}{1+x_1}\right) \frac{|x_2|}{(1+x_1)^2} \\ &= \frac{1}{2\pi} \exp\left\{ -\frac{1}{2}\left[\frac{x_1^2x_2^2}{(1+x_1)^2} + \frac{x_2^2}{(1+x_1)^2} \right] \right\} \frac{|x_2|}{(1+x_1)^2} \\ &= \frac{1}{2\pi} \exp\left\{ -\frac{1}{2} \frac{1+x_1^2}{(1+x_1)^2} x_2^2 \right\} \frac{|x_2|}{(1+x_1)^2} \end{aligned}$$

which has support $\mathbb{R}^2$. 5 Marks

   (b) *Find the covariance between random variables Y_1 and Y_2 where*

$$Y_1 = Z_1 + Z_2 \qquad\qquad Y_2 = Z_1 - Z_2.$$

We have by results from lectures that

$$\begin{aligned} \mathbb{E}_{Y_1}[Y_1] &= \mathbb{E}_{Z_1}[Z_1] + \mathbb{E}_{Z_2}[Z_2] = 0 \\ \mathbb{E}_{Y_2}[Y_2] &= \mathbb{E}_{Z_1}[Z_1] - \mathbb{E}_{Z_2}[Z_2] = 0 \end{aligned}$$

and therefore

$$\text{Cov}_{Y_1,Y_2}[Y_1,Y_2] = \mathbb{E}_{Y_1,Y_2}[Y_1Y_2] = \mathbb{E}_{Z_1,Z_2}[(Z_1+Z_2)(Z_1-Z_2)] = \mathbb{E}_{Z_1}[Z_1^2] - \mathbb{E}_{Z_2}[Z_2^2] = 0.$$

as Z_1 and Z_2 are identically distributed. 2 Marks

Are Y_1 and Y_2 independent ? Justify your answer. 1 Mark

We have that

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \mathbf{A} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

say, so therefore by properties of the multivariate normal, $(Y_1,Y_2)^\top \sim Normal_2(\mathbf{0},\Sigma)$, where

$$\Sigma = \mathbf{A}\mathbf{A}^\top = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and so as this matrix is diagonal, we have that Y_1 and Y_2 are independent.

(c) *Find the characteristic function of*

$$V = a_1 Z_1 + a_2 Z_2$$

for real constants a_1 *and* a_2.

We have

$$\begin{aligned}
\varphi_V(t) = \mathbb{E}_V[\exp\{itV\}] &= \mathbb{E}_{Z_1,Z_2}[\exp\{it(a_1Z_1 + a_2Z_2)\}] \\
&= \mathbb{E}_{Z_1}[\exp\{it(a_1Z_1)\}]\mathbb{E}_{Z_2}[\exp\{it(a_2Z_2)\}] \qquad \text{independence} \\
&= \varphi_{Z_1}(a_1t)\varphi_{Z_2}(a_2t) \\
&= \exp\left\{-\frac{a_1^2t^2}{2}\right\}\exp\left\{-\frac{a_2^2t^2}{2}\right\} \\
&= \exp\left\{-\frac{(a_1^2+a_2^2)t^2}{2}\right\}
\end{aligned}$$

2 Marks

2. *Suppose that* $X = (X_1, X_2)^\top \sim Dirichlet(\alpha_1, \alpha_2, \alpha_3)$ *where* $\alpha_1 = \alpha_2 = \alpha_3 = 2$.

(a) *Prove (showing your working) that marginally* $X_1 \sim Beta(a, b)$*, for* a, b *to be identified.*

The joint pdf is given from the handout as

$$f_{X_1,X_2}(x_1,x_2) = \frac{\Gamma(2+2+2)}{\Gamma(2)\Gamma(2)\Gamma(2)} x_1x_2(1-x_1-x_2)$$

on the simplex $\mathcal{S}_2$

$$\mathcal{S}_2 = \{(x_1,x_2) : 0 < x_1, x_2 < 1, 0 < x_1 + x_2 < 1\}.$$

and zero otherwise. For the marginal

$$f_{X_1}(x_1) = \int_0^{1-x_1} f_{X_1,X_2}(x_1,x_2)\, dx_2 \qquad 0 < x_1 < 1$$

as for any fixed x_1, $0 < x_1 + x_2 < 1$ implies that the joint pdf is only non-zero when $x_2 < 1 - x_1$. Thus, for $0 < x_1 < 1$,

$$\begin{aligned}
f_{X_1}(x_1) &= \frac{\Gamma(2+2+2)}{\Gamma(2)\Gamma(2)\Gamma(2)}\, x_1 \int_0^{1-x_1} x_2(1-x_1-x_2)\, dx_2 \\
&= \frac{\Gamma(2+2+2)}{\Gamma(2)\Gamma(2)\Gamma(2)}\, x_1 \int_0^1 (1-x_1)t(1-x_1-(1-x_1)t)\,(1-x_1)\, \mathrm{d}t \qquad t = x_2/(1-x_1) \\
&= \frac{\Gamma(2+2+2)}{\Gamma(2)\Gamma(2)\Gamma(2)}\, x_1(1-x_1)^3 \int_0^1 t(1-t)\, dt \\
&= \frac{\Gamma(2+2+2)}{\Gamma(2)\Gamma(2)\Gamma(2)}\, x_1(1-x_1)^3 \frac{\Gamma(2)\Gamma(2)}{\Gamma(2+2)} = \frac{\Gamma(2+2+2)}{\Gamma(2)\Gamma(4)}\, x_1(1-x_1)^3
\end{aligned}$$

so therefore $X_1 \sim Beta(2, 4)$. 3 Marks

(b) *Find the correlation between X_1 and V defined by*

$$V = 1 - X_1.$$

For the covariance

$$\mathbb{E}_{X_1,V}[X_1 V] \equiv \mathbb{E}_{X_1}[X_1(1-X_1)] = \mathbb{E}_{X_1}[X_1] - \mathbb{E}_{X_1}[X_1^2]$$

but

$$\mathbb{E}_{X_1}[X_1]\mathbb{E}_V[V] = \mathbb{E}_{X_1}[X_1](1-\mathbb{E}_{X_1}[X_1]) = \mathbb{E}_{X_1}[X_1] - \{\mathbb{E}_{X_1}[X_1]\}^2$$

so therefore

$$\text{Cov}_{X_1,V}[X_1,V] = \mathbb{E}_{X_1,V}[X_1V] - \mathbb{E}_{X_1}[X_1]\mathbb{E}_V[V] = \{\mathbb{E}_{X_1}[X_1]\}^2 - \mathbb{E}_{X_1}[X_1^2] = -\text{Var}_{X_1}[X_1].$$

Hence, as

$$\text{Var}_V[V] = \text{Var}_{X_1}[X_1]$$

we have that

$$\begin{aligned}\text{Corr}_{X_1,V}[X_1,V] &= \frac{\text{Cov}_{X_1,V}[X_1,V]}{\sqrt{\text{Var}_{X_1}[X_1]\text{Var}_V[V]}} \\ &= \frac{-\text{Var}_{X_1}[X_1]}{\sqrt{\text{Var}_{X_1}[X_1]\text{Var}_{X_1}[X_1]}} = -1.\end{aligned}$$

3 Marks

3. *Suppose that X and Y have joint distribution specified by*

$$X \sim Beta(1,1)$$
$$Y|X = x \sim Binomial(n,x)$$

for fixed $n \geq 1$. Find $Var_Y[Y]$.

For any rv

$$\mathbb{E}_X[X^2] = \{\mathbb{E}_X[X]\}^2 + \text{Var}_X[X]$$

and here

$$\mathbb{E}_X[X] = \frac{1}{2} \qquad \text{Var}_X[X] = \frac{1}{12} \qquad \mathbb{E}_X[X^2] = \frac{1}{3}.$$

Using iterated expectation, by properties of the Binomial distribution, we have

$$\mathbb{E}_Y[Y] = \mathbb{E}_X[\mathbb{E}_{Y|X}[Y|X]] = \mathbb{E}_X[nX] = n\mathbb{E}_X[X] = \frac{n}{2}$$

$$\mathbb{E}_Y[Y^2] = \mathbb{E}_X[\mathbb{E}_{Y|X}[Y^2|X]] = \mathbb{E}_X[n^2X^2 + nX(1-X)] = \frac{n(n-1)}{3} + \frac{n}{2}$$

so therefore

$$\text{Var}_Y[Y] = \frac{n(n-1)}{3} + \frac{n}{2} - \frac{n^2}{4} = \frac{4n(n-1)+6n-3n^2}{12} = \frac{n^2+2n}{12} = \frac{n(n+2)}{12}$$

4 Marks

Note also that

$$\begin{aligned}
\text{Var}_Y[Y] &= \mathbb{E}_X[\text{Var}_{Y|X}[Y|X]] + \text{Var}_X[\mathbb{E}_{Y|X}[Y|X]] \\
&= \mathbb{E}_X[nX(1-X)] + \text{Var}_X[nX] = n\mathbb{E}_X[X(1-X)] + n^2\text{Var}_X[X]
\end{aligned}$$

which may be computed using properties of the $Beta(1,1)$ distribution;

$$\mathbb{E}_X[X(1-X)] = \int_0^1 x(1-x)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

so

$$\text{Var}_Y[Y] = \frac{n}{6} + \frac{n^2}{12} = \frac{n(n+2)}{12}.$$

Finally, note that by direct computation, the marginal pmf is

$$\begin{aligned}
f_Y(y) &= \int_0^1 \binom{n}{y} x^y (1-x)^{n-y}\,dx \qquad\qquad y = 0, 1, \ldots, n \\
&= \binom{n}{y} \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)} \\
&= \frac{n!}{y!(n-y)!} \frac{y!}{(n-y)!} (n+1)! \\
&= \frac{1}{n+1}
\end{aligned}$$

and zero otherwise. Thus Y has a discrete Uniform distribution on the set $\{0, 1, \ldots, n\}$ and by direct calculation

$$\mathbb{E}_Y[Y] = \sum_{y=0}^{n} y \frac{1}{n+1} = \frac{1}{n+1} \sum_{y=0}^{n} y = \frac{1}{n+1} \frac{n(n+1)}{2} = \frac{n}{2}$$

and

$$\mathbb{E}_Y[Y^2] = \sum_{y=0}^{n} y^2 \frac{1}{n+1} = \frac{1}{n+1} \sum_{y=0}^{n} y^2 = \frac{1}{n+1} \frac{n(n+1)(2n+1)}{6} = \frac{n(2n+1)}{6}$$

and

$$\text{Var}_Y[Y] = \frac{n(2n+1)}{6} - \frac{n^2}{4} = \frac{n(n+2)}{12}.$$