MATH 556 - ASSIGNMENT 2 – SOLUTIONS

1. *Consider the discrete pmf, f_X, defined for $i = 1, 2, \ldots$ by*
$$f_X(x_i) = p_i$$
where $\mathbb{X} = \{x_1, x_2, \ldots\}$, with $x_i > 0$ for all i. Suppose that $\mu = \mathbb{E}_X[X] < \infty$.

   Show that
$$\mu e^{\mu} \leq \sum_{i=1}^{\infty} p_i x_i e^{x_i}.$$

   Apply Jensen's inequality to the function $g(x) = xe^x$ – this function is convex on $\mathbb{R}^+$, as
$$\frac{d^2}{dx^2}\{g(x)\} = (x+2)e^x > 0 \quad \text{for } x > 0.$$
   Therefore
$$\mu e^{\mu} = g\left(\mathbb{E}_X[X]\right) \leq \mathbb{E}_X[g(X)] = \sum_{i=1}^{\infty} p_i x_i e^{x_i}.$$

   4 Marks

2. *For the following cfs, φ_X, find the corresponding distribution (by name, or in terms of the pmf, pdf or cdf), or demonstrate why the function is not a valid cf. You may quote results from the distributions formula sheet or from lectures.*

   (a) *For $t \in \mathbb{R}$*
$$\varphi_X(t) = \frac{2}{2+t^2}$$
   This is the cf of a scaled version of the Double Exponential pdf
$$\varphi_X(t) = \frac{2}{2+t^2} = \frac{1}{1+(t/\sqrt{2})^2} = \varphi_Z(t/\sqrt{2})$$
   where $Z \sim DE(1)$. Therefore $X \stackrel{d}{=} Z/\sqrt{2}$ and
$$f_X(x) = \frac{1}{2\sqrt{2}} \exp\left\{-\frac{|x|}{\sqrt{2}}\right\} \qquad z \in \mathbb{R}$$

   2 Marks

   (b) *For $t \in \mathbb{R}$*
$$\varphi_X(t) = \frac{1}{2}(1 + \cos(t) + i\sin(t))$$
   We have that
$$\varphi_X(t) = \frac{e^{it0} + e^{it1}}{2}$$
   so X is uniformly distributed on $\{0, 1\}$, with $P_X[X=0] = P_X[X=1] = 1/2$. 2 Marks

   (c) *For $t \in \mathbb{R}$*
$$\varphi_X(t) = \frac{1}{2}e^{it}\left(1 + \exp\{e^{it} - 1 - it\}\right)$$
   We have
$$\varphi_X(t) = \frac{1}{2}\left(e^{it} + \exp\{(e^{it} - 1)\}\right)$$
   so X is distributed as a equal mixture of a mass at $x = 1$ and a $Poisson(1)$, 4 Marks

3. *A key result for cfs is that if X and Y are independent, and $Z = X + Y$, then*

$$\varphi_Z(t) = \varphi_X(t)\varphi_Y(t).$$

*Does this result ever hold if $Z = X + Y$ but X and Y are **not** independent? Justify your answer.*

Yes it can hold even if X and Y are not independent. For example, if we take $Y = X$ (with probability 1), the variables are certainly not independent, and $Z = X + X = 2X$. By the linear transformation result

$$\varphi_Z(t) = \varphi_X(2t).$$

We require that $\varphi_X(2t) = \{\varphi_X(t)\}^2$. This is satisfied by the cf

$$\varphi_X(t) = \exp\{-|t|\}.$$

4 Marks

4. *Suppose that $X \sim Exponential(1)$ with cf denoted $\varphi_X(t)$, and that $\phi(\cdot)$ is the standard normal pdf. Consider the function*

$$\varphi(t) = \int_{-\infty}^{\infty} \varphi_X(ts)\phi(s)\, ds.$$

Is $\varphi(t)$ a valid cf ? Justify your answer.

Several potential methods using sufficient conditions. More directly, we have that

$$\begin{aligned}
\varphi(t) = \int_{-\infty}^{\infty} \varphi_X(ts)\phi(s)\, ds &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{itsx} f_X(x)\, dx \right\} \phi(s)\, ds && \text{definition of } \varphi_X \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{itsx} \phi(s) f_X(x)\, dx\, ds \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{itv} \frac{1}{x} \phi(v/x) f_X(x)\, dv\, dx && v = sx \\
&= \int_{-\infty}^{\infty} e^{itv} \left\{ \int_{-\infty}^{\infty} \frac{1}{x}\, \phi(v/x)\, f_X(x)\, dx \right\} dv \\
&= \int_{-\infty}^{\infty} e^{itv} f_V(v)\, dv
\end{aligned}$$

say, where

$$f_V(v) = \int_{-\infty}^{\infty} \frac{1}{x}\, \phi(v/x)\, f_X(x)\, dx$$

This is the marginal pdf arising from the joint model on (X, V) constructed as

$$X \sim Exponential(1) \qquad V|X = x \sim Normal(0, 1/x^2).$$

so $\varphi(t)$ is the cf for this marginal. 4 Marks

Note: these calculations would work for any pair of distributions not merely the Exponential and Normal.

Alternatively,

$$\begin{aligned}
\varphi(t) &= \mathbb{E}_Y\left[\varphi_X(tY)\right] && Y \sim Normal(0,1) \\
&= \mathbb{E}_Y\left[\mathbb{E}_{X|Y}[\exp\{itYX\}|Y]\right] \\
&= \mathbb{E}_{X,Y}[\exp\{itXY\}] \\
&\equiv \mathbb{E}_Z[\exp\{itZ\}]
\end{aligned}$$

say, where $Z = XY$. Hence $\varphi(t) \equiv \varphi_Z(t)$.