Name: ID:

McGill University

Faculty of Science

Final Examination

MATH 556: Mathematical Statistics I

Examiner: Professor J. Nešlehová

Associate Examiner: Professor D.B. Wolfson

Date: Thursday, December 18, 2013

Time: 9:00 A.M. – 12:00 P.M.

Instructions

- **This is a closed book exam.**
- **The exam comprises one title page, three pages of questions and two pages of formulas.**
- **Answer all six questions in the examination booklets provided.**
- **Calculators and translation dictionaries are permitted.**
- **A formula sheet is provided.**

Good Luck!

Problem 1

The Fisher–Snedecor F_{ν_1,ν_2} distribution with parameters $\nu_1 > 0$ and $\nu_2 > 0$ has density

$$f(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} x^{\nu_1/2-1} \left(1 + \frac{\nu_1 x}{\nu_2}\right)^{-(\nu_1+\nu_2)/2}$$

whenever $x > 0$. Suppose that X is a random variable with the F_{ν_1,ν_2} distribution.

(a) Compute the expectation of X. What can you say about the moment generating function of X? **(5 Marks)**

(b) Compute $\operatorname{corr}(X, 1/X)$ and list three drawbacks of Pearson's correlation coefficient. You can use, without proof, that $\operatorname{var}(X) = \{2\nu_2^2(\nu_1 + \nu_2 - 2)\}/\{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)\}$. **(4 Marks)**

(c) Prove that the random variable

$$Y = \frac{\nu_1 X}{\nu_2 + \nu_1 X}$$

has a $\text{Beta}(\nu_1/2, \nu_2/2)$ distribution. **(5 Marks)**

(d) Suppose that Y is as in part (c) with $\nu_1 = 2$. Determine a transformation h so that $h(Y)$ is Geometric$(1/2)$ as given on the formula sheet. State all results that you use. **(5 Marks)**

(e) Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two random samples, each from respective univariate distributions and let S_n^2 and T_m^2, respectively, denote the corresponding sample variances. State all conditions under which S_n^2/T_m^2 has an $F_{n-1,m-1}$ distribution. **(3 Marks)**

Problem 2

Suppose that the random pair (X, Y) has a bivariate Normal distribution with density given by

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\varrho^2}} \exp\left\{-\frac{x^2 + y^2 - 2\varrho xy}{2(1-\varrho^2)}\right\}$$

for all $x, y \in \mathbb{R}$ and some $\varrho \in (-1, 1)$. Let also $W = X^2$ and $Z = Y^2$.

(a) Show that the joint density of (W, Z) is given, for $w, z > 0$, by

$$\frac{1}{4\pi\sqrt{(1-\varrho^2)wz}} \left\{1 + \exp\left(-\frac{2\varrho\sqrt{wz}}{1-\varrho^2}\right)\right\} \exp\left\{-\frac{w + z - 2\varrho\sqrt{wz}}{2(1-\varrho^2)}\right\}.$$

(5 Marks)

(b) Determine the conditional density of Z given $W = w$. **(5 Marks)**

(c) Determine the joint distribution of (U, V), where

$$U = Z + W, \quad V = \frac{Z}{Z + W}.$$

Under which condition are U and V independent? Which well-known distributions do they have in this case? **(5 Marks)**

Problem 3

(a) Consider the following family of densities with parameters $\sigma > 0$ and $\mu \in \mathbb{R}$:

$$f(x;\mu,\sigma) = \frac{1}{x\sqrt{2\pi}\sigma}\exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}, \quad x > 0.$$

- (i) Show that the family $\{f(\cdot;\mu,\sigma)\}$ constitutes an exponential family. **(2 Marks)**
- (ii) Determine the natural parametrization and the natural parameter space. **(3 Marks)**
- (iii) Compute $\mathrm{E}(\ln X)^3$ when X has density $f(\cdot;\mu,\sigma)$. **(5 Marks)**

(b) Explain how a new family of densities can be constructed from a given density g by exponential tilting. Can this approach be used when $g(x) = f(x;\mu,\sigma)$ from part (a) with some given values of μ and σ? If yes, give the tilted family, if not, explain why. **(5 Marks)**

(c) Give an example of a family of distributions that is not an exponential family. **(3 Marks)**

Problem 4

The logarithmic series (LS) distribution is a discrete distribution with parameter $p \in (0,1)$ and probability mass function

$$f(x) = -\frac{1}{\ln(p)}\frac{(1-p)^x}{x}, \quad x \in \{1,2,3,\dots\}.$$

(a) Derive the moment generating function of the $\mathrm{LS}(p)$ distribution and compute the mean and variance of $X \sim \mathrm{LS}(p)$. **(5 Marks)**

(b) Consider the following two-level hierarchical model:

$$N \sim \text{Poisson}(\lambda), \quad \lambda > 0$$
$$S|N = n \sim X_1 + \cdots + X_n,$$

where $X_1,\dots,X_n$ are i.i.d. with $\mathrm{LS}(p)$ distribution, $p \in (0,1)$. Compute the mean and variance of the marginal (unconditional) distribution of S. **(5 Marks)**

(c) Determine the marginal (unconditional) distribution of S. **(5 Marks)**

(d) Prove that for any two variables X and Y with finite variances, X and $Y - \mathrm{E}(Y|X)$ are uncorrelated. **(5 Marks)**

Problem 5

(a) Suppose that X and Y are random variables such that $\mathrm{E}|X|^p < \infty$ and $\mathrm{E}|Y|^p < \infty$ for some $p \geq 1$. Using any result shown in class, prove the so-called Minkowski inequality, viz.

$$(\mathrm{E}|X+Y|^p)^{1/p} \leq (\mathrm{E}|X|^p)^{1/p} + (\mathrm{E}|Y|^p)^{1/p}.$$

(5 Marks)

(b) Suppose that X and Y are Normal(μ, σ^2) random variables that are not necessarily independent or jointly Normal. Show that for any $x > 0$,

$$\Pr(X + Y \geq x) \leq \frac{4(\sigma^2 + \mu^2)}{x^2}.$$

(5 Marks)

Problem 6

Let $X_1, \ldots, X_n$ be a random sample of size $n \geq 2$ from the uniform distribution on the interval $(0, \theta)$. When θ is unknown, it can be estimated by the "maximum likelihood estimator"

$$\hat{\theta}_n = \max(X_1, \ldots, X_n).$$

(a) Show that $\hat{\theta}_n$ converges in probability to θ as $n \to \infty$; estimators that have this property are said to be "consistent." **(5 Marks)**

(b) Show that as $n \to \infty$, $n(\theta - \hat{\theta}_n)$ converges in distribution to an exponential random variable with mean θ. **(5 Marks)**

(c) A differentiable function g is said to be a "variance stabilizing transformation" whenever the limiting variance of $g(\hat{\theta}_n)$ does not depend on θ. Identify this transformation and compute the limiting distribution of $n\{g(\theta) - g(\hat{\theta}_n)\}$. **(5 Marks)**

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(r,p)$	$\{0,1,2,...\}$	$r \in \mathbb{Z}^+, p \in (0,1)$	$\binom{r+x-1}{x}p^r(1-p)^x$		$\dfrac{r(1-p)}{p}$	$\dfrac{r(1-p)}{p^2}$	$\left(\dfrac{p}{1-e^t(1-p)}\right)^r$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx, \quad \alpha > 0$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathrm{E}_{f_Y}[Y] = \mu+\sigma\mathrm{E}_{f_X}[X] \qquad \mathrm{var}_{f_Y}[Y] = \sigma^2\mathrm{var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathrm{E}_{f_X}[X]$	$\mathrm{var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard: $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\frac{1}{\beta-\alpha}$	$\frac{x-\alpha}{\beta-\alpha}$	$\frac{(\alpha+\beta)}{2}$	$\frac{(\beta-\alpha)^2}{12}$	$\frac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard: $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard: $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(\frac{\beta}{\beta-t}\right)^\alpha$
$Normal(\mu,\sigma^2)$ (standard: $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
χ^2_ν	$\mathbb{R}^+$	$\nu\in\mathbb{N}$	$\frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\nu/2}}x^{(\nu/2)-1}e^{-x/2}$		ν	2ν	$(1-2t)^{-\nu/2}$
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\frac{\theta}{\theta+x}\right)^\alpha$	$\frac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\frac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	