McGill University

Course:	MATH 556
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Faculty of Science

December 2007

MATH 556

MATHEMATICAL STATISTICS I

Setter	Checker	Associate Examiner
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McGill University

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MATH 556

MATHEMATICAL STATISTICS I

Final Examination

Date: 14th December 2007 Time: 2pm-5pm

Examiner: Prof. D. A. Stephens Associate Examiner: Prof. R. J. Steele

This paper contains six questions.

Credit will be given for all questions attempted.

Calculators may not be used. A Formula Sheet is provided.

1. Suppose for all of question 1 that continuous random variable X has a $Uniform(0,1)$ distribution.

(a) Find the probability density function (pdf) of random variable Y defined by

$$Y = \log\left(\frac{X}{1-X}\right).$$

Find also the expectation of Y.

6 MARKS

(b) Find the pdf of Z where

$$Z = X(1-X)$$

Find also the expectation of Z.

6 MARKS

(c) Suppose that X_1 and X_2 are independent, and have the same distribution as X. Find the probability

$$\Pr\left[X_1X_2 > \frac{1}{2}\right]$$

and the probability

$$\Pr\left[(1-X_1)(1-X_2) > \frac{1}{2}\right]$$

8 MARKS

2. Suppose that Z_1 and Z_2 are independent random variables each having a $Normal(0,1)$ distribution.

(a) Find the joint pdf of random variables X_1 and X_2 defined by

$$X_1 = \frac{Z_1}{Z_2} \qquad X_2 = Z_1 + Z_2.$$

8 MARKS

(b) Find the marginal distribution of X_1.

4 MARKS

(c) Find the covariance between random variables Y_1 and Y_2 where

$$Y_1 = Z_1^2 \qquad Y_2 = Z_1^3$$

4 MARKS

(d) Find the moment generating function of

$$V = \alpha Z_1 + \beta Z_2$$

for real constants α and β.

4 MARKS

Show your working in all cases.

3. (a) Suppose that X has pdf

$$f_X(x) = \frac{1}{2\sigma} \exp\{-|x/\sigma|\} \qquad -\infty < x < \infty$$

for parameter $\sigma > 0$. Find the characteristic function of X.

8 MARKS

(b) Suppose that $X_1, \ldots, X_n$ are independent and identically distributed random variables with characteristic function

$$C_X(t) = \exp\{-|t|^\alpha\} \qquad t \in \mathbb{R}$$

for parameter $0 < \alpha \leq 2$.

(i) Are $X_1, \ldots, X_n$ continuous random variables ? Justify your answer.

2 MARKS

(ii) Is the distribution of $X_1, \ldots, X_n$ infinitely divisible ? Justify your answer.

4 MARKS

(iii) Find real constants a_n and b_n such that T_n defined by

$$T_n = a_n + b_n \sum_{i=1}^{n} X_i$$

has the same distribution as X_1.

6 MARKS

4. (a) Suppose that X is a random variable, with mgf $M_X(t)$ defined on a neighbourhood $(-h, h)$ of zero. Let a be a real value. Show that

$$P[X \geq a] \leq e^{-at} M_X(t) \qquad \text{for } 0 < t < h.$$

You must prove explicitly every step that you use.

10 MARKS

(b) State and prove Minkowski's Inequality for random variables X and Y.

You may quote without proof Hölder's Inequality: if X and Y are two random variables, and $p, q > 1$ satisfy

$$p^{-1} + q^{-1} = 1$$

then

$$|E_{f_{X,Y}}[XY]| \leq E_{f_{X,Y}}[|XY|] \leq \{E_{f_X}[|X|^p]\}^{1/p} \{E_{f_Y}[|Y|^q]\}^{1/q}$$

10 MARKS

5. (a) (i) Write down the form of an *Exponential Family distribution* in its *natural* (or *canonical*) parameterization.

5 MARKS

(ii) Suppose that X has a one-parameter, natural Exponential Family distribution with natural parameter η, and pmf/pdf $f_X(x|\eta)$. Show that

$$E_{f_X}[X] = \kappa(\eta)$$

for some function κ to be identified.

You may quote without proof properties of the score function.

5 MARKS

(iii) Suppose that $X \sim Gamma(\alpha, 1)$. Is the distribution of $Y = 1/X$ an Exponential Family distribution ? If so, find the natural parameter. If not, explain why not.

4 MARKS

(b) Consider the three-level hierarchical model:

LEVEL 3 :	$\mu \in \mathbb{R}, \tau, \sigma > 0$	Fixed parameters
LEVEL 2 :	$M \sim Normal(\mu, \tau^2)$	
LEVEL 1 :	$X_1, \ldots, X_n \mid M = m \sim Normal(m, \sigma^2)$	

where $X_1, \ldots, X_n$ are mutually conditionally independent given M, denoted $X_i \perp X_j \mid M$, for all i, j. Find the (marginal) variance-covariance matrix of the n-dimensional vector random variable $\underset{\sim}{X} = (X_1, \ldots, X_n)^\mathsf{T}$.

6 MARKS

6. (a) Suppose that random variable X_n has cdf

$$F_{X_n}(x) = \left(\frac{n\lambda x}{1 + n\lambda x} \right)^n \qquad 0 < x < \infty$$

and zero otherwise, for parameter $\lambda > 0$. If it exists, find the limiting distribution of X_n as $n \longrightarrow \infty$.

6 MARKS

(b) Suppose $X_1, \ldots, X_n$ are a random sample from a distribution with cdf F_X given by

$$F_X(x) = 1 - x^{-1} \qquad x \geq 1$$

and zero otherwise. Show that $Z_n = \min\{X_1, \ldots, X_n\}$ has a degenerate limiting distribution as $n \longrightarrow \infty$, but that $U_n = (Z_n)^{\alpha_n}$ has the same distribution as X_n for some real value α_n.

8 MARKS

(c) Find an approximation to the distribution of the random variable

$$T_n = \exp\{-1/\overline{X}_n\}$$

where $\overline{X}_n$ is the mean of a random sample from an $Exponential(\lambda)$ distribution, and n is large.

6 MARKS

DISCRETE DISTRIBUTIONS

	RANGE $\mathbb{X}$	PARAMETERS	MASS FUNCTION f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Bernoulli(\theta)$	$\{0,1\}$	$\theta \in (0,1)$	$\theta^x(1-\theta)^{1-x}$		θ	$\theta(1-\theta)$	$1-\theta+\theta e^t$
$Binomial(n,\theta)$	$\{0,1,...,n\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n}{x}\theta^x(1-\theta)^{n-x}$		$n\theta$	$n\theta(1-\theta)$	$(1-\theta+\theta e^t)^n$
$Poisson(\lambda)$	$\{0,1,2,...\}$	$\lambda \in \mathbb{R}^+$	$\dfrac{e^{-\lambda}\lambda^x}{x!}$		λ	λ	$\exp\{\lambda(e^t-1)\}$
$Geometric(\theta)$	$\{1,2,...\}$	$\theta \in (0,1)$	$(1-\theta)^{x-1}\theta$	$1-(1-\theta)^x$	$\dfrac{1}{\theta}$	$\dfrac{(1-\theta)}{\theta^2}$	$\dfrac{\theta e^t}{1-e^t(1-\theta)}$
$NegBinomial(n,\theta)$	$\{n,n+1,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{x-1}{n-1}\theta^n(1-\theta)^{x-n}$		$\dfrac{n}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta e^t}{1-e^t(1-\theta)}\right)^n$
or	$\{0,1,2,...\}$	$n \in \mathbb{Z}^+, \theta \in (0,1)$	$\binom{n+x-1}{x}\theta^n(1-\theta)^x$		$\dfrac{n(1-\theta)}{\theta}$	$\dfrac{n(1-\theta)}{\theta^2}$	$\left(\dfrac{\theta}{1-e^t(1-\theta)}\right)^n$

For **CONTINUOUS** distributions (see over), define the **GAMMA FUNCTION**

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

and the LOCATION/SCALE transformation $Y = \mu + \sigma X$ gives

$$f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right)\frac{1}{\sigma} \qquad F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right) \qquad M_Y(t) = e^{\mu t}M_X(\sigma t) \qquad \mathsf{E}_{f_Y}[Y] = \mu + \sigma\mathsf{E}_{f_X}[X] \qquad \mathsf{Var}_{f_Y}[Y] = \sigma^2\mathsf{Var}_{f_X}[X]$$

CONTINUOUS DISTRIBUTIONS

	$\mathbb{X}$	PARAMS.	PDF f_X	CDF F_X	$\mathsf{E}_{f_X}[X]$	$\mathsf{Var}_{f_X}[X]$	MGF M_X
$Uniform(\alpha,\beta)$ (standard model $\alpha=0,\beta=1$)	(α,β)	$\alpha<\beta\in\mathbb{R}$	$\dfrac{1}{\beta-\alpha}$	$\dfrac{x-\alpha}{\beta-\alpha}$	$\dfrac{(\alpha+\beta)}{2}$	$\dfrac{(\beta-\alpha)^2}{12}$	$\dfrac{e^{\beta t}-e^{\alpha t}}{t(\beta-\alpha)}$
$Exponential(\lambda)$ (standard model $\lambda=1$)	$\mathbb{R}^+$	$\lambda\in\mathbb{R}^+$	$\lambda e^{-\lambda x}$	$1-e^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$	$\left(\dfrac{\lambda}{\lambda-t}\right)$
$Gamma(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$		$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$	$\left(\dfrac{\beta}{\beta-t}\right)^\alpha$
$Weibull(\alpha,\beta)$ (standard model $\beta=1$)	$\mathbb{R}^+$	$\alpha,\beta\in\mathbb{R}^+$	$\alpha\beta x^{\alpha-1}e^{-\beta x^\alpha}$	$1-e^{-\beta x^\alpha}$	$\dfrac{\Gamma(1+1/\alpha)}{\beta^{1/\alpha}}$	$\dfrac{\Gamma(1+2/\alpha)-\Gamma(1+1/\alpha)^2}{\beta^{2/\alpha}}$	
$Normal(\mu,\sigma^2)$ (standard model $\mu=0,\sigma=1$)	$\mathbb{R}$	$\mu\in\mathbb{R},\sigma\in\mathbb{R}^+$	$\dfrac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$		μ	σ^2	$e^{\{\mu t+\sigma^2t^2/2\}}$
$Student(\nu)$	$\mathbb{R}$	$\nu\in\mathbb{R}^+$	$\dfrac{(\pi\nu)^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\left\{1+\frac{x^2}{\nu}\right\}^{(\nu+1)/2}}$		0 (if $\nu>1$)	$\dfrac{\nu}{\nu-2}$ (if $\nu>2$)	
$Pareto(\theta,\alpha)$	$\mathbb{R}^+$	$\theta,\alpha\in\mathbb{R}^+$	$\dfrac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$1-\left(\dfrac{\theta}{\theta+x}\right)^\alpha$	$\dfrac{\theta}{\alpha-1}$ (if $\alpha>1$)	$\dfrac{\alpha\theta^2}{(\alpha-1)(\alpha-2)}$ (if $\alpha>2$)	
$Beta(\alpha,\beta)$	$(0,1)$	$\alpha,\beta\in\mathbb{R}^+$	$\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$		$\dfrac{\alpha}{\alpha+\beta}$	$\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	