# 556: MATHEMATICAL STATISTICS I

## STOCHASTIC CONVERGENCE

The following definitions relate to random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. The statements are given in terms of $P$ (rather than $P_X$ etc) for simplicity.

- **Convergence in Probability:** The sequence of random variables $X_1, \ldots, X_n$ **converges in probability** to constant $c$, denoted
$$X_n \xrightarrow{p} c$$
if
$$\lim_{n \longrightarrow \infty} P\left[|X_n - c| < \epsilon\right] = 1 \qquad \text{or} \qquad \lim_{n \longrightarrow \infty} P\left[|X_n - c| \geq \epsilon\right] = 0$$
that is, if the limiting distribution of $X_1, \ldots, X_n$ is **degenerate at** $c$.

  **Weak Law Of Large Numbers:** Suppose that $X_1, \ldots, X_n$ is a sequence of i.i.d. random variables with expectation $\mu$ and finite variance $\sigma^2$. Let $\overline{X}_n$ be the sample mean. Then, for all $\epsilon > 0$,
$$\lim_{n \longrightarrow \infty} P\left[\left|\overline{X}_n - \mu\right| < \epsilon\right] = 1,$$
that is, $\overline{X}_n \xrightarrow{p} \mu$, and thus the mean of $X_1, \ldots, X_n$ converges in probability to $\mu$. We have seen that $\overline{X}_n$ has expectation $\mu$ and variance $\sigma^2/n$, and hence by the Chebychev Inequality,
$$P\left[\left|\overline{X}_n - \mu\right| \geq \epsilon\right] \leq \frac{\sigma^2}{n\epsilon^2} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$
for all $\epsilon > 0$. Hence $\overline{X}_n \xrightarrow{p} \mu$ as
$$P\left[\left|\overline{X}_n - \mu\right| < \epsilon\right] \longrightarrow 1 \qquad \text{as } n \longrightarrow \infty.$$

  The sequence of random variables $X_1, \ldots, X_n$ **converges in probability** to random variable $X$
$$X_n \xrightarrow{p} X$$
if, for all $\epsilon > 0$,
$$\lim_{n \longrightarrow \infty} P\left[|X_n - X| < \epsilon\right] = 1 \qquad \text{or equivalently} \qquad \lim_{n \longrightarrow \infty} P\left[|X_n - X| \geq \epsilon\right] = 0$$
To understand this definition, let $\epsilon > 0$, and consider
$$A_n(\epsilon) \equiv \{\omega : |X_n(\omega) - X(\omega)| \geq \epsilon\}$$
Then we have $X_n \xrightarrow{p} X$ if
$$\lim_{n \longrightarrow \infty} P(A_n(\epsilon)) = 0$$
that is, if there exists an $n$ such that for all $m \geq n$, $P(A_m(\epsilon)) < \epsilon$.

- **Convergence Almost Surely:** The sequence of random variables $X_1, \ldots, X_n$ **converges almost surely** to random variable $X$, denoted $X_n \xrightarrow{a.s.} X$ if for every $\epsilon > 0$
$$P\left[\lim_{n \longrightarrow \infty} |X_n - X| < \epsilon\right] = 1,$$

that is, if $A \equiv \{\omega : X_n(\omega) \longrightarrow X(\omega)\}$, then $P(A) = 1$. Equivalently, $X_n \stackrel{a.s.}{\longrightarrow} X$ if for every $\epsilon > 0$

$$P\left[\lim_{n\longrightarrow\infty} |X_n - X| \geq \epsilon\right] = 0.$$

This can also be written

$$\lim_{n\longrightarrow\infty} X_n(\omega) = X(\omega)$$

for every $\omega \in \Omega$, except possibly those lying in a set of probability zero under $P$.

**Alternative characterizations**:

(I) Let $\epsilon > 0$, and the sets $A_m(\epsilon)$ and $B_n(\epsilon)$ be defined for $n, m \geq 1$ by

$$A_m(\epsilon) \equiv \{\omega : |X_m(\omega) - X(\omega)| \geq \epsilon\} \qquad\qquad B_n(\epsilon) \equiv \bigcup_{m=n}^{\infty} A_m(\epsilon).$$

Then $X_n \stackrel{a.s.}{\longrightarrow} X$ if and only if $P(B_n(\epsilon)) \longrightarrow 0$ as $n \longrightarrow \infty$.

- $A_m(\epsilon)$ is the set of $\omega$ for which $X_m(\omega)$ is at least $\epsilon$ away from $X$.
- $B_n(\epsilon)$ is the set of $\omega$ for which $X_m(\omega)$ at least $\epsilon$ away from $X$, for **at least one** $m \geq n$.
- The event $B_n(\epsilon)$ occurs **if there exists** an $m \geq n$ such that $|X_m - X| \geq \epsilon$.
- $X_n \stackrel{a.s.}{\longrightarrow} X$ if and only if and only if $P(B_m(\epsilon)) \longrightarrow 0$.

(II) $X_n \stackrel{a.s.}{\longrightarrow} X$ if and only if

$$P[\,|X_n - X| \geq \epsilon \text{ infinitely often}\,] = 0$$

that is, $X_n \stackrel{a.s.}{\longrightarrow} X$ if and only if there are **only finitely many** $X_n$ for which $|X_n(\omega) - X(\omega)| \geq \epsilon$ if $\omega$ lies in a set of probability greater than zero.

Note that $X_n \stackrel{a.s.}{\longrightarrow} X$ if and only if

$$\lim_{n\longrightarrow\infty} P(B_n(\epsilon)) = \lim_{n\longrightarrow\infty} P\left(\bigcup_{m=n}^{\infty} A_m(\epsilon)\right) = 0$$

in contrast with the definition of convergence in probability, where $X_n \stackrel{p}{\longrightarrow} X$ if

$$\lim_{n\longrightarrow\infty} P(A_n(\epsilon)) = 0.$$

Clearly

$$A_n(\epsilon) \subseteq \bigcup_{m=n}^{\infty} A_m(\epsilon)$$

so therefore

$$\lim_{n\longrightarrow\infty} P\left(\bigcup_{m=n}^{\infty} A_m(\epsilon)\right) = 0 \qquad \Longrightarrow \qquad \lim_{n\longrightarrow\infty} P(A_n(\epsilon)) = 0$$

and hence almost sure convergence implies convergence in probability.

**Alternative terminology**:

- $X_n \longrightarrow X$ *almost everywhere*, $X_n \stackrel{a.e.}{\longrightarrow} X$
- $X_n \longrightarrow X$ *with probability 1*, $X_n \stackrel{w.p.1}{\longrightarrow} X$

**Interpretation:** A random variable is a real-valued function from (a sigma-algebra defined on) sample space $\Omega$ to $\mathbb{R}$ . The sequence of random variables $X_1, \ldots, X_n$ corresponds to a sequence of functions defined on elements of $\Omega$. Almost sure convergence requires that the sequence of real numbers $X_n(\omega)$ converges to $X(\omega)$ (as a real sequence) for all $\omega \in \Omega$, as $n \longrightarrow \infty$, except perhaps when $\omega$ is in a set having probability zero under the probability distribution of $X$.

**Strong Law Of Large Numbers:** Suppose that $X_1, \ldots, X_n$ is a sequence of i.i.d. random variables with expectation $\mu$ and (finite) variance $\sigma^2$. Let $\overline{X}_n$ be the sample mean. Then for all $\epsilon > 0$,

$$P\left[\lim_{n\longrightarrow\infty} \left|\overline{X}_n - \mu\right| < \epsilon\right] = 1,$$

that is, $\overline{X}_n \xrightarrow{a.s.} \mu$, and thus the mean of $X_1, \ldots, X_n$ converges almost surely to $\mu$.

- **Convergence in Distribution:** Consider a sequence of random variables $X_1, X_2, \ldots$ and a corresponding sequence of cdfs, $F_{X_1}, F_{X_2}, \ldots$ so that for $n = 1, 2, ..$ $F_{X_n}(x) = P\left[X_n \leq x\right]$. Suppose that there exists a cdf, $F_X$, such that **for all** $x$ **at which** $F_X$ **is continuous**,

$$\lim_{n\longrightarrow\infty} F_{X_n}(x) = F_X(x).$$

Then $X_1, \ldots, X_n$ **converges in distribution** to random variable $X$ with cdf $F_X$, denoted

$$X_n \xrightarrow{d} X$$

and $F_X$ is the **limiting distribution**. Convergence of a sequence of mgfs or cfs also indicates convergence in distribution, that is, if for all $t$ at which $M_X(t)$ is defined, if as $n \longrightarrow \infty$, we have

$$M_{X_i}(t) \longrightarrow M_X(t) \qquad \Longleftrightarrow \qquad X_n \xrightarrow{d} X.$$

The sequence of random variables $X_1, \ldots, X_n$ converges in distribution to constant $c$ if the limiting distribution of $X_1, \ldots, X_n$ is **degenerate at** $c$, that is,

$$X_n \xrightarrow{d} X$$

and $P\left[X = c\right] = 1$, so that

$$F_X(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$$

**Interpretation:** This special case of convergence in distribution occurs when the limiting distribution is discrete, with the probability mass function only being non-zero at a single value, that is, if the limiting random variable is $X$, then $P[X = c] = 1$ and zero otherwise. We say that the sequence of random variables $X_1, \ldots, X_n$ **converges in distribution** to $c$ if and only if, for all $\epsilon > 0$,

$$\lim_{n\longrightarrow\infty} P\left[|X_n - c| < \epsilon\right] = 1$$

This definition indicates that convergence in distribution to a constant $c$ occurs if and only if the probability becomes increasingly concentrated around $c$ as $n \longrightarrow \infty$.

To show that we should ignore points of discontinuity of $F_X$ in the definition of convergence in distribution, consider the following example: let

$$F_\epsilon(x) = \begin{cases} 0 & x < \epsilon \\ 1 & x \geq \epsilon \end{cases}$$

be the cdf of a degenerate distribution with probability mass 1 at $x = \epsilon$. Now consider a sequence $\{\epsilon_n\}$ of real values converging to $\epsilon$ from **below**. Then, as $\epsilon_n < \epsilon$, we have

$$F_{\epsilon_n}(x) = \begin{cases} 0 & x < \epsilon_n \\ 1 & x \geq \epsilon_n \end{cases}$$

which converges to $F_\epsilon(x)$ at all real values of $x$. However, if instead $\{\epsilon_n\}$ converges to $\epsilon$ from **above**, then $F_{\epsilon_n}(\epsilon) = 0$ for each finite $n$, as $\epsilon_n > \epsilon$, so $\lim\limits_{n \longrightarrow \infty} F_{\epsilon_n}(\epsilon) = 0$. Hence, as $n \longrightarrow \infty$,

$$F_{\epsilon_n}(\epsilon) \longrightarrow 0 \neq 1 = F_\epsilon(\epsilon).$$

Thus the limiting function in this case is

$$F_\epsilon(x) = \begin{cases} 0 & x \leq \epsilon \\ 1 & x > \epsilon \end{cases}$$

which is not a cdf as it is not right-continuous. However, if $\{X_n\}$ and $X$ are random variables with distributions $\{F_{\epsilon_n}\}$ and $F_\epsilon$, then $P[X_n = \epsilon_n] = 1$ converges to $P[X = \epsilon] = 1$, however we take the limit, so $F_\epsilon$ does describe the limiting distribution of the sequence $\{F_{\epsilon_n}\}$. Thus, because of right-continuity, we ignore points of discontinuity in the limiting function.

- **Convergence In $r$th Mean**
  The sequence of random variables $X_1, \ldots, X_n$ **converges in rth mean** to random variable $X$, denoted $X_n \xrightarrow{r} X$ if
  $$\lim_{n \longrightarrow \infty} \mathbb{E}\left[|X_n - X|^r\right] = 0.$$
  For example, if
  $$\lim_{n \longrightarrow \infty} \mathbb{E}\left[(X_n - X)^2\right] = 0$$
  then we write $X_n \xrightarrow{r=2} X$. In this case, we say that $\{X_n\}$ converges to $X$ *in mean-square* or *in quadratic mean*. For $r_1 > r_2 \geq 1$,
  $$X_n \xrightarrow{r=r_1} X \qquad \Longrightarrow \qquad X_n \xrightarrow{r=r_2} X$$
  as, by Lyapunov's inequality
  $$\mathbb{E}[\,|X_n - X|^{r_2}\,]^{1/r_2} \leq \mathbb{E}[\,|X_n - X|^{r_1}\,]^{1/r_1} \qquad \therefore \qquad \mathbb{E}[\,|X_n - X|^{r_2}\,] \leq \mathbb{E}[\,|X_n - X|^{r_1}\,]^{r_2/r_1} \longrightarrow 0$$
  as $n \longrightarrow \infty$, as $r_2 < r_1$. Thus
  $$\mathbb{E}[\,|X_n - X|^{r_2}\,] \longrightarrow 0$$
  and $X_n \xrightarrow{r=r_2} X$. The converse does not hold in general.

- **Relating The Modes Of Convergence** For sequence of random variables $X_1, \ldots, X_n$,
  $$\left.\begin{array}{c} X_n \xrightarrow{a.s.} X \\ \text{or} \\ X_n \xrightarrow{r} X \end{array}\right\} \quad \Longrightarrow \quad X_n \xrightarrow{p} X \quad \Longrightarrow \quad X_n \xrightarrow{d} X$$
  so almost sure convergence and convergence in $r$th mean for some $r$ both imply convergence in probability, which in turn implies convergence in distribution to random variable $X$.

  No other relationships hold in general

- **Slutsky's Theorem:** Suppose that

$$X_n \xrightarrow{d} X \qquad \text{and} \qquad Y_n \xrightarrow{p} c$$

Then

(i) $X_n + Y_n \xrightarrow{d} X + c$

(ii) $X_n Y_n \xrightarrow{d} cX$

(iii) $X_n / Y_n \xrightarrow{d} X/c$ provided $c \neq 0$.

- **The Central Limit Theorem:** Suppose $X_1, \ldots, X_n$ are i.i.d. random variables with mgf $M_X$, with expectation $\mu$ and variance $\sigma^2$, both finite. Let the random variable $Z_n$ be defined by

$$Z_n = \frac{\sum\limits_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}} = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}$$

and denote by $M_{Z_n}$ the mgf of $Z_n$. Then, as $n \longrightarrow \infty$,

$$M_{Z_n}(t) \longrightarrow \exp\{t^2/2\}$$

**irrespective** of the form of $M_X$. Thus, as $n \longrightarrow \infty$, $Z_n \xrightarrow{d} Z \sim Normal(0,1)$.

*Proof.* First, let $Y_i = (X_i - \mu)/\sigma$ for $i = 1, \ldots, n$. Then $Y_1, \ldots, Y_n$ are i.i.d. with mgf $M_Y$ say, and $\mathbb{E}_{f_Y}[Y_i] = 0$, $\text{Var}_Y[Y_i] = 1$ for each $i$. Using a Taylor series expansion, we have that for $t$ in a neighbourhood of zero,

$$M_Y(t) = 1 + t\mathbb{E}_Y[Y] + \frac{t^2}{2!}\mathbb{E}_Y[Y^2] + \frac{t^3}{3!}\mathbb{E}_Y[Y^3] + \ldots = 1 + \frac{t^2}{2} + \mathrm{O}(t^3)$$

using the $\mathrm{O}(t^3)$ notation to capture all terms involving $t^3$ and higher powers. Re-writing $Z_n$ as

$$Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i$$

as $Y_1, \ldots, Y_n$ are independent, we have by a standard mgf result that

$$M_{Z_n}(t) = \prod_{i=1}^{n} \left\{ M_Y\left(\frac{t}{\sqrt{n}}\right) \right\} = \left\{ 1 + \frac{t^2}{2n} + \mathrm{O}(n^{-3/2}) \right\}^n = \left\{ 1 + \frac{t^2}{2n} + \mathrm{o}(n^{-1}) \right\}^n.$$

so that, by the definition of the exponential function, as $n \longrightarrow \infty$

$$M_{Z_n}(t) \longrightarrow \exp\{t^2/2\} \qquad \therefore \qquad Z_n \xrightarrow{d} Z \sim Normal(0,1)$$

where no further assumptions on $M_X$ are required. ∎

**Alternative statement:** The theorem can also be stated in terms of

$$Z_n = \frac{\sum\limits_{i=1}^{n} X_i - n\mu}{\sqrt{n}} = \sqrt{n}(\overline{X}_n - \mu)$$

so that

$$Z_n \xrightarrow{d} Z \sim Normal(0, \sigma^2).$$

and $\sigma^2$ is termed the **asymptotic variance** of $Z_n$.

**Notes :**

(i) The theorem requires the **existence of the mgf** $M_X$.

(ii) The theorem holds for the i.i.d. case, but there are similar theorems for **non identically distributed**, and **dependent** random variables.

(iii) The theorem allows the construction of **asymptotic normal approximations**. For example, for **large but finite** $n$, by using the properties of the Normal distribution,

$$\begin{aligned} \overline{X}_n &\sim \mathcal{AN}(\mu, \sigma^2/n) \\ S_n = \sum_{i=1}^{n} X_i &\sim \mathcal{AN}(n\mu, n\sigma^2). \end{aligned}$$

where $\mathcal{AN}(\mu, \sigma^2)$ denotes an asymptotic normal distribution. The notation

$$\overline{X}_n \stackrel{\cdot}{\sim} Normal(\mu, \sigma^2/n)$$

is sometimes used.

(iv) The **multivariate version** of this theorem can be stated as follows: Suppose $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are i.i.d. $d$-dimensional random variables with mgf $M_{\mathbf{X}}$, with

$$\mathbb{E}_{\mathbf{X}}[\mathbf{X}_i] = \boldsymbol{\mu} \qquad \text{Var}_{\mathbf{X}}[\mathbf{X}_i] = \Sigma$$

where $\Sigma$ is a positive definite, symmetric $k \times k$ matrix defining the variance-covariance matrix of the $\mathbf{X}_i$. Let the random variable $\mathbf{Z}_n$ be defined by

$$\mathbf{Z}_n = \sqrt{n}(\overline{\mathbf{X}}_n - \boldsymbol{\mu})$$

where

$$\overline{\mathbf{X}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i.$$

Then

$$\mathbf{Z}_n \xrightarrow{d} \mathbf{Z} \sim Normal_d(\mathbf{0}, \Sigma)$$

as $n \longrightarrow \infty$.

## Appendix: Technical Details

*Proof.* **Relating the modes of convergence.**

(a) $X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{p} X$. Suppose $X_n \xrightarrow{a.s.} X$, and let $\epsilon > 0$. Then

$$P[\, |X_n - X| < \epsilon \,] \geq P[\, |X_m - X| < \epsilon, \ \forall m \geq n \,] \tag{1}$$

as, considering the original sample space,

$$\{\omega : |X_m(\omega) - X(\omega)| < \epsilon, \ \forall m \geq n\} \subseteq \{\omega : |X_n(\omega) - X(\omega)| < \epsilon\}$$

But, as $X_n \xrightarrow{a.s.} X$, $P[\, |X_m - X| < \epsilon, \ \forall m \geq n \,] \longrightarrow 1$, as $n \longrightarrow \infty$. So, after taking limits in equation (1), we have

$$\lim_{n \longrightarrow \infty} P[\, |X_n - X| < \epsilon \,] \geq \lim_{n \longrightarrow \infty} P[\, |X_m - X| < \epsilon, \ \forall m \geq n \,] = 1$$

and so

$$\lim_{n \longrightarrow \infty} P[\, |X_n - X| < \epsilon \,] = 1 \qquad \therefore \qquad X_n \xrightarrow{p} X.$$

(b) $X_n \xrightarrow{r} X \implies X_n \xrightarrow{p} X$. Suppose $X_n \xrightarrow{r} X$, and let $\epsilon > 0$. Then, using an argument similar to Chebychev's Lemma,

$$\mathbb{E}[\, |X_n - X|^r \,] \geq \mathbb{E}[\, |X_n - X|^r I_{\{|X_n - X| > \epsilon\}} \,] \geq \epsilon^r P[|X_n - X| > \epsilon].$$

Taking limits as $n \longrightarrow \infty$, as $X_n \xrightarrow{r} X$, $\mathbb{E}[\, |X_n - X|^r \,] \longrightarrow 0$ as $n \longrightarrow \infty$, so therefore

$$P[|X_n - X| > \epsilon] \longrightarrow 0 \qquad \therefore \qquad X_n \xrightarrow{p} X.$$

(c) $X_n \xrightarrow{p} X \implies X_n \xrightarrow{d} X$. Suppose $X_n \xrightarrow{p} X$, and let $\epsilon > 0$. Denote, in the usual way,

$$F_{X_n}(x) = P[X_n \leq x] \qquad \text{and} \qquad F_X(x) = P[X \leq x].$$

Then, by the theorem of total probability, we have two inequalities

$$F_{X_n}(x) = P[X_n \leq x] = P[X_n \leq x, X \leq x+\epsilon] + P[X_n \leq x, X > x+\epsilon] \leq F_X(x+\epsilon) + P[|X_n - X| > \epsilon]$$

$$F_X(x-\epsilon) = P[X \leq x-\epsilon] = P[X \leq x-\epsilon, X_n \leq x] + P[X \leq x-\epsilon, X_n > x] \leq F_{X_n}(x) + P[|X_n - X| > \epsilon].$$

as $A \subseteq B \implies P(A) \leq P(B)$ yields

$$P[\, X_n \leq x, X \leq x+\epsilon \,] \leq F_X(x+\epsilon) \qquad \text{and} \qquad P[\, X \leq x-\epsilon, X_n \leq x \,] \leq F_{X_n}(x).$$

Thus

$$F_X(x-\epsilon) - P[\, |X_n - X| > \epsilon] \leq F_{X_n}(x) \leq F_X(x+\epsilon) + P[\, |X_n - X| > \epsilon]$$

and taking limits as $n \longrightarrow \infty$ (with care; we cannot yet write $\lim_{n \longrightarrow \infty} F_{X_n}(x)$ as we do not know that this limit exists) recalling that $X_n \xrightarrow{p} X$,

$$F_X(x-\epsilon) \leq \liminf_{n \longrightarrow \infty} F_{X_n}(x) \leq \limsup_{n \longrightarrow \infty} F_{X_n}(x) \leq F_X(x+\epsilon)$$

Then if $F_X$ is continuous at $x$, $F_X(x-\epsilon) \longrightarrow F_X(x)$ and $F_X(x+\epsilon) \longrightarrow F_X(x)$ as $\epsilon \longrightarrow 0$, so

$$F_X(x) \leq \liminf_{n \longrightarrow \infty} F_{X_n}(x) \leq \limsup_{n \longrightarrow \infty} F_{X_n}(x) \leq F_X(x)$$

and thus $F_{X_n}(x) \longrightarrow F_X(x)$ as $n \longrightarrow \infty$.

Thus all results follow. ∎

**THEOREM (Partial Converses)**

(i) If

$$\sum_{n=1}^{\infty} P[\,|X_n - X| > \epsilon\,] < \infty$$

for every $\epsilon > 0$, then $X_n \xrightarrow{a.s.} X$.

(ii) If, for some positive integer $r$,

$$\sum_{n=1}^{\infty} \mathbb{E}[\,|X_n - X|^r\,] < \infty$$

then $X_n \xrightarrow{a.s.} X$.

*Proof.* (i) Let $\epsilon > 0$. Then for $n \geq 1$,

$$P[\,|X_n - X| > \epsilon, \text{ for some } m \geq n\,] \equiv P\left[\bigcup_{m=n}^{\infty} \{|X_m - X| > \epsilon\}\right] \leq \sum_{m=n}^{\infty} P[\,|X_m - X| > \epsilon\,]$$

as, by elementary probability theory, $P(A \cup B) \leq P(A) + P(B)$. But, as it is the tail sum of a convergent series (by assumption), it follows that

$$\lim_{n \longrightarrow \infty} \sum_{m=n}^{\infty} P[\,|X_m - X| > \epsilon\,] = 0.$$

Hence

$$\lim_{n \longrightarrow \infty} P[\,|X_n - X| > \epsilon, \text{ for some } m \geq n\,] = 0$$

and $X_n \xrightarrow{a.s.} X$.

(ii) Identical to part (i), and using part (b) of the previous theorem that $X_n \xrightarrow{r} X \Longrightarrow X_n \xrightarrow{p} X$.

Thus the partial converse results hold. ∎