556: MATHEMATICAL STATISTICS I

MULTIVARIATE 1-1 TRANSFORMATIONS

We consider the case of 1-1 transformations g, as in this case the probability transform result coincides with changing variables in a d-dimensional integral. We can consider $g = (g_1, \ldots, g_d)$ as a vector of functions forming the components of the new random vector $\mathbf{Y}$.

Given a collection of variables $(X_1, \ldots, X_d)$ with support $\mathbb{X}^{(d)}$ and joint pdf $f_{X_1,\ldots,X_d}$ we can construct the pdf of a transformed set of variables $(Y_1, \ldots, Y_d)$ using the following steps:

1. Write down the set of transformation functions $g_1, \ldots, g_d$

$$\begin{aligned} Y_1 &= g_1(X_1, \ldots, X_d) \\ &\vdots \\ Y_d &= g_d(X_1, \ldots, X_d) \end{aligned} .$$

2. Write down the set of inverse transformation functions $g_1^{-1}, \ldots, g_d^{-1}$

$$\begin{aligned} X_1 &= g_1^{-1}(Y_1, \ldots, Y_d) \\ &\vdots \\ X_d &= g_d^{-1}(Y_1, \ldots, Y_d) \end{aligned}$$

3. Consider the joint support of the new variables, $\mathbb{Y}^{(k)}$.

4. Compute the Jacobian of the transformation: first form the matrix of partial derivatives

$$D_y = \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} & \cdots & \dfrac{\partial x_1}{\partial y_d} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} & \cdots & \dfrac{\partial x_2}{\partial y_d} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial x_d}{\partial y_1} & \dfrac{\partial x_d}{\partial y_2} & \cdots & \dfrac{\partial x_d}{\partial y_d} \end{bmatrix}$$

where, for each (i, j)

$$\frac{\partial x_i}{\partial y_j} = \frac{\partial}{\partial y_j}\left\{g_i^{-1}(y_1, \ldots, y_d)\right\}$$

and then set $|J(y_1, \ldots, y_d)| = |\det D_y|$

Note that

$$\det D_y = \det D_y^\top$$

so that an alternative but equivalent Jacobian calculation can be carried out by forming $D_y^\top$. *Note also that*

$$|J(y_1, \ldots, y_d)| = \frac{1}{|J(x_1, \ldots, x_d)|}$$

where $J(x_1, \ldots, x_d)$ *is the Jacobian of the transformation regarded in the reverse direction (that is, if we start with* $(Y_1, \ldots, Y_d)$ *and transfrom to* $(X_1, \ldots, X_d)$*)*

5. Write down the joint pdf of $(Y_1, \ldots, Y_d)$ as

$$f_{Y_1,\ldots,Y_d}(y_1, \ldots, y_d) = f_{X_1,\ldots,X_d}\left(g_1^{-1}(y_1, \ldots, y_d), \ldots, g_d^{-1}(y_1, \ldots, y_d)\right) \times |J(y_1, \ldots, y_d)|$$

for $(y_1, \ldots, y_d) \in \mathbb{Y}^{(k)}$

EXAMPLE Suppose that X_1 and X_2 have joint pdf

$$f_{X_1,X_2}(x_1,x_2) = 2 \qquad 0 < x_1 < x_2 < 1$$

and zero otherwise. Compute the joint pdf of random variables

$$Y_1 = \frac{X_1}{X_2} \qquad\qquad Y_2 = X_2$$

SOLUTION

1. Given that $\mathbb{X}^{(2)} \equiv \{(x_1,x_2) : 0 < x_1 < x_2 < 1\}$ and

$$g_1(t_1,t_2) = \frac{t_1}{t_2} \qquad\qquad g_2(t_1,t_2) = t_2$$

2. Inverse transformations:

$$\left.\begin{array}{l} Y_1 = X_1/X_2 \\ Y_2 = X_2 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} X_1 = Y_1Y_2 \\ X_2 = Y_2 \end{array}\right.$$

and thus

$$g_1^{-1}(t_1,t_2) = t_1t_2 \qquad\qquad g_2^{-1}(t_1,t_2) = t_2$$

3. Range: to find $\mathbb{Y}^{(2)}$ consider point by point transformation from $\mathbb{X}^{(2)}$ to $\mathbb{Y}^{(2)}$ For a pair of points $(x_1,x_2) \in \mathbb{X}^{(2)}$ and $(y_1,y_2) \in \mathbb{Y}^{(2)}$ linked via the transformation, we have

$$0 < x_1 < x_2 < 1 \Longleftrightarrow 0 < y_1y_2 < y_2 < 1$$

and hence we can extract the inequalities

$$0 < y_2 < 1 \text{ and } 0 < y_1 < 1 \qquad\qquad \mathbb{Y}^{(2)} \equiv (0,1) \times (0,1)$$

4. The Jacobian for points $(y_1,y_2) \in \mathbb{Y}^{(2)}$ is

$$D_y = \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix} \Rightarrow |J(y_1,y_2)| = |\det D_y| = |y_2| = y_2$$

Note that for points $(x_1,x_2) \in \mathbb{X}^{(2)}$ is

$$D_x = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{x_2} & \dfrac{x_1}{x_2^2} \\ 0 & 1 \end{bmatrix} \Rightarrow |J(x_1,x_2)| = |\det D_x| = \left|\frac{1}{x_2}\right| = \frac{1}{x_2}$$

so that

$$|J(y_1,y_2)| = \frac{1}{|J(x_1,x_2)|}$$

5. Finally, we have

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1y_2,y_2) \times y_2 = 2y_2 \qquad 0 < y_1 < 1, 0 < y_2 < 1$$

and zero otherwise

EXAMPLE Suppose that X_1 and X_2 are **independent** and **identically distributed** random variables defined on $\mathbb{R}^+$ each with pdf of the form

$$f_X(x) = \sqrt{\frac{1}{2\pi x}} \exp\left\{-\frac{x}{2}\right\} \qquad x > 0$$

and zero otherwise. Compute the joint pdf of random variables $Y_1 = X_1$ and $Y_2 = X_1 + X_2$

SOLUTION

1. Given that $\mathbb{X}^{(2)} \equiv \{(x_1, x_2) : 0 < x_1, 0 < x_2\}$ and

$$g_1(t_1, t_2) = t_1 \qquad\qquad g_2(t_1, t_2) = t_1 + t_2$$

2. Inverse transformations:

$$\left.\begin{array}{l} Y_1 = X_1 \\ Y_2 = X_1 + X_2 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} X_1 = Y_1 \\ X_2 = Y_2 - Y_1 \end{array}\right.$$

and thus

$$g_1^{-1}(t_1, t_2) = t_1 \qquad\qquad g_2^{-1}(t_1, t_2) = t_2 - t_1$$

3. Range: to find $\mathbb{Y}^{(2)}$ consider point by point transformation from $\mathbb{X}^{(2)}$ to $\mathbb{Y}^{(2)}$ For a pair of points $(x_1, x_2) \in \mathbb{X}^{(2)}$ and $(y_1, y_2) \in \mathbb{Y}^{(2)}$ linked via the transformation; as both original variables are strictly positive, we can extract the inequalities

$$0 < y_1 < y_2 < \infty$$

4. The Jacobian for points $(y_1, y_2) \in \mathbb{Y}^{(2)}$ is

$$D_y = \begin{bmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \Rightarrow |J(y_1, y_2)| = |\det D_y| = |1| = 1$$

Note, here, $J(x_1, x_2) = |\det D_x| = 1$ also so that again

$$|J(y_1, y_2)| = \frac{1}{|J(x_1, x_2)|}$$

5. Finally, we have for $0 < y_1 < y_2 < \infty$

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(y_1, y_2 - y_1) \times 1 = f_{X_1}(y_1) \times f_{X_2}(y_2 - y_1) \qquad \text{by independence} \\ &= \sqrt{\frac{1}{2\pi y_1}} \exp\left\{-\frac{y_1}{2}\right\} \sqrt{\frac{1}{2\pi (y_2 - y_1)}} \exp\left\{-\frac{(y_2 - y_1)}{2}\right\} \\ &= \frac{1}{2\pi} \frac{1}{\sqrt{y_1 (y_2 - y_1)}} \exp\left\{-\frac{y_2}{2}\right\} \end{aligned}$$

and zero otherwise

Here, for $y_2 > 0$

$$\begin{aligned}
f_{Y_2}(y_2) &= \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_1 = \int_0^{y_2} \frac{1}{2\pi}\frac{1}{\sqrt{y_1(y_2-y_1)}}\exp\left\{-\frac{y_2}{2}\right\}\,dy_1 \\
&= \frac{1}{2\pi}\exp\left\{-\frac{y_2}{2}\right\}\int_0^{y_2}\frac{1}{\sqrt{y_1(y_2-y_1)}}\,dy_1 \\
&= \frac{1}{2\pi}\exp\left\{-\frac{y_2}{2}\right\}\int_0^{1}\frac{1}{\sqrt{ty_2(y_2-ty_2)}}y_2\,dt \qquad \text{setting } y_1 = ty_2 \\
&= \frac{1}{2\pi}\exp\left\{-\frac{y_2}{2}\right\}\int_0^{1}\frac{1}{\sqrt{t(1-t)}}\,dt \\
&= \frac{1}{2}\exp\left\{-\frac{y_2}{2}\right\}
\end{aligned}$$

as

$$\int_0^1 \frac{1}{\sqrt{t(1-t)}}\,dt = \pi$$

either by direct calculation, or by recognizing the integrand as proportional to a $Beta(1/2,1/2)$ pdf.

Special Case: Convolution
Suppose that X_1 and X_2 have a joint pmf or pdf, f_{X_1,X_2}, and let $Y = X_1 + X_2$. We compute the pmf/pdf of Y by using a Convolution Formula, which for continuous variables is a special case of the transformation theorem.

- **Discrete Case:** By the Theorem of Total Probability, we have from first principles that for any fixed y.

$$f_Y(y) = P_Y[Y=y] = \sum_{x_1}\sum_{\substack{x_2 \\ x_1+x_2=y}} f_{X_1,X_2}(x_1,x_2) = \sum_{x_1} f_{X_1,X_2}(x_1, y-x_1)$$

- **Continuous Case:** Consider $Y = X_1 + X_2$ and $Z = X_1$. We have

$$\left.\begin{array}{l} Y = X_1 + X_2 \\ Z = X_1 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} X_1 = Z \\ X_2 = Y - Z \end{array}\right.$$

The Jacobian of this transform is 1, so we conclude from the transformation result that for all (y,z)

$$f_{Y,Z}(y,z) = f_{X_1,X_2}(z, y-z)$$

and hence, marginalizing z, we see that

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)\,dz = \int_{-\infty}^{\infty} f_{X_1,X_2}(z,y-z)\,dz$$

which we may rewrite

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, y-x_1)\,dx_1.$$

NOTES:

1. It is important to record the support of the new variable Y when recording the form of f_Y.
2. The marginalization over x_1 must take into account the support of f_{X_1,X_2}: that is, for any fixed y only contributions to the sum or integral where

$$f_{X_1,X_2}(x_1, y-x_1) > 0.$$