556: MATHEMATICAL STATISTICS I

THE CHARACTERISTIC FUNCTION INVERSION FORMULA

The general inversion formula defines how F_X can be computed from φ_X. Let $\overline{F}_X(x)$ be defined by

$$\overline{F}_X(x) = \frac{1}{2}\left\{F_X(x) + \lim_{y \longrightarrow x^-} F_X(y)\right\}.$$

Then for $a < b$

$$\overline{F}_X(b) - \overline{F}_X(a) = \frac{1}{2\pi}\lim_{T \longrightarrow \infty}\int_{-T}^{T}\left(\frac{e^{-iat} - e^{-ibt}}{it}\right)\varphi_X(t)\,dt.$$

Note first that

$$\frac{e^{-iat} - e^{-ibt}}{it} = \int_a^b e^{-ity}\,dy.$$

Therefore

$$\begin{aligned}
\int_{-T}^{T}\left(\frac{e^{-iat} - e^{-ibt}}{it}\right)\varphi_X(t)\,dt &= \int_{-T}^{T}\left\{\int_a^b e^{-ity}\,dy\right\}\left\{\int_{-\infty}^{\infty} e^{itx} dF_X(x)\right\}\,dt \\
&= \int_{-T}^{T}\int_a^b\int_{-\infty}^{\infty} e^{-ity}e^{itx} dF_X(x)\,dy\,dt \qquad \text{(exchanging the order of intgn.)} \\
&= \int_{-\infty}^{\infty}\int_{-T}^{T}\int_a^b e^{-it(y-x)}\,dy\,dt\,dF_X(x) \\
&= \int_{-\infty}^{\infty}\int_{-T}^{T}\frac{1}{it}\left(e^{-it(a-x)} - e^{-it(b-x)}\right)\,dt\,dF_X(x). \qquad (1)
\end{aligned}$$

Define

$$g_1(a,b,T,x) = \int_{-T}^{T}\frac{1}{it}\left(e^{-it(a-x)} - e^{-it(b-x)}\right)\,dt \qquad g_2(T,c) = \int_0^T \frac{\sin(ct)}{t}\,dt$$

so that, as $\cos$ and $\sin$ are even and odd functions about zero respectively, we have that

$$g_1(a,b,T,x) = 2\int_0^T\left(\frac{\sin((a-x)t)}{t} - \frac{\sin((b-x)t)}{t}\right) = 2g_2(T,a-x) - 2g_2(T,b-x).$$

We have (see `https://en.wikipedia.org/wiki/Dirichlet_integral`) that

$$\lim_{T \longrightarrow \infty} g_2(T,c) = \int_0^\infty \frac{\sin(ct)}{t}\,dt = \begin{cases} \dfrac{\pi}{2} & c > 0 \\ 0 & c = 0 \\ -\dfrac{\pi}{2} & c < 0 \end{cases}$$

so therefore for any fixed a, b, x, considering the possible signs of $a - x$ and $b - x$, we have

$$\lim_{T \longrightarrow \infty} g_1(a,b,T,x) = \begin{cases} 0 & x < a \text{ or } x > b \\ \pi & x = a \text{ or } x = b \\ 2\pi & a < x < b. \end{cases}$$

Because of this, and because $g_2(T,1)$ is continuous in T, we can deduce that $|g_1(a,b,T,x)|$ is bounded, and hence (by dominated convergence) we can pass the limit under the integral in equation (1), that is

$$\begin{aligned}\lim_{T\longrightarrow\infty}\int_{-T}^{T}\left(\frac{e^{-iat}-e^{-ibt}}{it}\right)\varphi_X(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\longrightarrow\infty} g_1(a,b,T,x)dF_X(x)\\ &= \overline{F}_X(b)-\overline{F}_X(a)\end{aligned}$$

using the definition of $\overline{F}_X(x)$