MATH 556 - MID-TERM SOLUTIONS 2008

1. (a) From first principles (univariate transformation theorem also acceptable): for $z \in (0, 1/4)$

$$F_Z(z) = P_Z\left[Z \le z\right] = P_X\left[X(1-X) \le z\right] = P_X\left[X \le x_1(z) \cap X \ge x_2(z)\right]$$

where $x_1(z)$ and $x_2(z)$ are the roots of the quadratic $x^2 - x + z = 0$, that is

$$x_1(z) = \frac{1-\sqrt{1-4z}}{2} \qquad\qquad x_2(z) = \frac{1+\sqrt{1-4z}}{2}.$$

Hence

$$F_Z(z) = 1 - \sqrt{1-4z} \qquad 0 < z < 1/4.$$

and therefore

$$f_Y(y) = \frac{2}{\sqrt{1-4z}} \qquad 0 < z < 1/4$$

and zero otherwise. For the expectation, using the Beta integral

$$\mathbb{E}_{f_Z}[Z] = \mathbb{E}_{f_X}[X(1-X)] = \int_0^1 x(1-x)\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

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(b)

$$P_{X_1,X_2}\left[X_1X_2 > \frac{1}{2}\right] = \int_{1/2}^1 \int_{1/(2x_1)}^1 dx_2\,dx_1 = \int_{1/2}^1 (1 - 1/(2x_1))\,dx_1 = \left[x - \frac{1}{2}\log x_1\right]_{1/2}^1$$

Hence

$$P\left[X_1X_2 > \frac{1}{2}\right] = \left(1 - \frac{1}{2}\log 1\right) - \left(\frac{1}{2} - \frac{1}{2}\log\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2}\log 2$$

As the distributions of X_1 and $1 - X_1$ are identical, we also have

$$P\left[(1-X_1)(1-X_2) > \frac{1}{2}\right] = \frac{1}{2} - \frac{1}{2}\log 2$$

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2. (a) By properties of the multivariate normal, $\underset{\sim}{X} \sim \mathcal{N}(\underset{\sim}{0}, \Sigma)$, where

$$\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$$

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(b) The covariance between random variables Y_1 and Y_2 is

$$Cov_{f_{Y_1,Y_2}}[Y_1, Y_2] = \mathbb{E}_{f_{Y_1,Y_2}}[Y_1Y_2] - \mathbb{E}_{f_{Y_1}}[Y_1]\mathbb{E}_{f_{Y_2}}[Y_2] \equiv \mathbb{E}_{f_{Z_1}}[Z_1^5] - \mathbb{E}_{f_{Z_1}}[Z_1^2]\mathbb{E}_{f_{Z_1}}[Z_1^3] = 0$$

as the odd moments of the standard normal are zero.

6 MARKS

(c) Find the mgf of V is

$$M_V(t) = \mathbb{E}_{f_V}[e^{tV}] = \mathbb{E}_{f_{Z_1,Z_2}}[\exp\{t(\alpha Z_1 + \beta Z_2)\}] = M_{Z_1}(\alpha t)M_{Z_2}(\beta t) = \exp\{(\alpha^2+\beta^2)t^2/2\}$$

3 MARKS

3. (a) By inspection

$$C_X(t) = \mathbb{E}_{f_X}[e^{itX}] = \frac{1}{2\sigma}\int_{-\infty}^{\infty} e^{itx}\lambda e^{-|x/\sigma|}\,dx$$

But f_X is symmetric about zero, so

$$C_X(t) = \frac{1}{\sigma}\int_0^{\infty}\cos(tx)e^{-x/\sigma}\,dx = \int_0^{\infty}\cos(sy)e^{-y}\,dy$$

where $s = \sigma t$, after changing from x to $y = x/\sigma$. Integrating by parts yields

$$C_X(t) = \frac{1}{1+\sigma^2 t^2}$$

as

$$\begin{aligned} C_X(s) &= \int_0^{\infty}\cos(sy)e^{-y}\,dy = \left[-\cos(sy)e^{-y}\right]_0^{\infty} - \int_0^{\infty} s\sin(sy)e^{-y}\,dy \\ &= 1 - s\left[\sin(sy)e^{-y}\right]_0^{\infty} - s\int_0^{\infty} s\cos(sy)e^{-y}\,dy = 1 - s^2 C_X(s) \end{aligned}$$

6 MARKS

(b) (i) $X_1,\ldots,X_n$ are continuous random variables, as $|C_X(t)| \longrightarrow 0$ as $t \longrightarrow \infty$

4 MARKS

(ii) We have by elementary cf results that

$$C_{T_n}(t) = e^{a_n it}\{C_X(b_n t)\}^n = e^{a_n it}\left\{\exp\{-n|2b_n t|^{\alpha}\}\right\} = e^{a_n it}\left\{\exp\{-n|b_n|^{\alpha}|2t|^{\alpha}\}\right\}$$

Thus we must have $a_n = 0$ (as $C_X(t)$ is entirely real) and

$$b_n = n^{-1/\alpha}$$

5 MARKS

4. (a) A general Exponential Family in canonical parameterization takes the form

$$f_X(x|\underset{\sim}{\eta}) = h(x)c^{\star}(\underset{\sim}{\eta})\exp\left\{\sum_{j=1}^{k}\eta_j t_j(x)\right\}$$

where $\underset{\sim}{\eta} = (\eta_1,\ldots,\eta_k)^{\mathsf{T}}$ is the natural parameter vector. *4 MARKS*

(b) A natural Exponential Family has $k = 1$ and takes the form

$$f_X(x|\eta) = h(x)c^{\star}(\eta)\exp\{\eta x\}$$

where η is the natural parameter. Let $S(X;\eta)$ be defined by

$$S(X;\eta) = \frac{d}{d\eta}\log f_X(X;\eta) = \frac{d}{d\eta}\{\log c^{\star}(\eta)\} + X$$

This is the score function, and we know that $\mathbb{E}_{f_X}[S(X;\eta)] = 0$, so therefore

$$0 = \frac{d}{d\eta}\{\log c^{\star}(\eta)\} + \mathbb{E}_{f_X}[X] \qquad \therefore \qquad \mathbb{E}_{f_X}[X] = -\frac{d}{d\eta}\{\log c^{\star}(\eta)\}$$

6 MARKS

(c) By the univariate transformation theorem

$$f_Y(y|\alpha) = \frac{1}{\Gamma(\alpha)}\left(\frac{1}{y}\right)^{\alpha+1}\exp\left\{-\frac{1}{y}\right\} \qquad x > 0$$

Thus, if $\eta = -(\alpha+1)$, we have for $x \in \mathbb{R}$

$$f_Y(y|\eta) = I_{(0,\infty)}(y)\exp\left\{-\frac{1}{y}\right\}\frac{1}{\Gamma(-1-\eta)}\exp\{\eta\log y\}$$

so this is an Exponential Family distribution with natural parameter $\eta = -(\alpha+1)$.

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