MATH 556 - MID-TERM SOLUTIONS

1. (a) (i) From first principles (univariate transformation theorem also acceptable): for $0 < x < 1$

$$F_X(x) = P[X \leq x] = P[\sin(\pi U/2) \leq x] = P\left[U \leq \frac{2}{\pi}\arcsin x\right] = \frac{2}{\pi}\arcsin x$$

and zero otherwise, as the sine function is monotonic increasing on $(0, \pi/2)$. Thus,

$$f_X(x) = \frac{2}{\pi\sqrt{1-x^2}} \qquad 0 < x < 1$$

and zero otherwise.

6 MARKS

(ii) We have by direct calculation

$$E_{f_X}[X] = \int_0^1 x \frac{2}{\pi\sqrt{1-x^2}}\,dx = \int_0^1 \sin(\pi u/2)\,du = \left[-\frac{2}{\pi}\cos(\pi u/2)\right]_0^1 = \frac{2}{\pi}.$$

4 MARKS

(iii) The area of the triangle is $A = U^2/2$, so the expected area of the triangle is

$$E_{f_A}[A] = E_{f_U}[U^2/2] = \int_0^1 u^2/2\,du = \frac{1}{6}.$$

6 MARKS

(b) We have from the formula sheet

$$f_{Y,Z}(y,z) = f_{Y|Z}(y|z)f_Z(z) = \frac{1}{\sqrt{z}}\frac{\lambda^{3/2}}{\Gamma(3/2)}z^{3/2-1}e^{-\lambda z} = \frac{\lambda^{3/2}}{\Gamma(3/2)}e^{-\lambda z} \qquad 0 < y < \sqrt{z} < \infty$$

Hence

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y,z)\,dz = \int_{y^2}^{\infty}\frac{\lambda^{3/2}}{\Gamma(3/2)}e^{-\lambda z}\,dz = \frac{\lambda^{1/2}}{\Gamma(3/2)}\exp\{-\lambda y^2\} \qquad y > 0$$

and zero otherwise.

9 MARKS

2. (a) Given $R = r$ with $0 < r < 1$, we require that,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y|R}(x,y|r)dx\,dy = 1 \qquad\qquad \int_{y=-r}^{y=r}\left\{\int_{x=-\sqrt{r^2-y^2}}^{x=\sqrt{r^2-y^2}} k(r)\,dx\right\}dy = 1.$$

The conditional density is **constant** on the disk radius r centered at the origin, which has area πr^2. Therefore we must have

$$k(r) = \frac{1}{\pi r^2} \qquad 0 < r < 1$$

8 MARKS

(b) The full joint pdf is therefore

$$f_{R,X,Y}(r,x,y) = f_{X,Y|R}(x,y|r) f_R(r) = \frac{1}{\pi r^2} \, 4r^3 = \frac{4r}{\pi}$$

on the region defined by

$$-r < x < r, \ -r < y < r, \ 0 < x^2 + y^2 < r^2, \ 0 < r < 1,$$

and zero otherwise. To get the joint marginal for X and Y, we integrate out R from the full joint pdf, that is

$$f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f_{R,X,Y}(r,x,y) \, dr \quad = \quad \int_{\sqrt{x^2+y^2}}^{1} \frac{4r}{\pi} \, dr = \frac{1}{\pi} \left[2r^2 \right]_{\sqrt{x^2+y^2}}^{1} = \frac{2}{\pi} \left[1 - x^2 - y^2 \right]$$

on the region defined by $0 < x < 1, \ 0 < y < 1, \ 0 < x^2 + y^2 < 1$ (that is, the unit circle) and zero otherwise.

9 MARKS

(c) The joint pdf is symmetric in form in x and y, and has support that is the unit circle. The joint pdf is also even in both x and y, and therefore $E_{f_X}[X] = E_{f_Y}[Y] = 0$, and also

$$E_{f_{X,Y}}[XY] = \int_{y=-1}^{y=1} \left\{ \int_{x=-\sqrt{1^2-y^2}}^{x=\sqrt{1^2-y^2}} xy \left(1 - x^2 - y^2 \right) \, dx \right\} \, dy = 0$$

Thus the covariance is zero.

4 MARKS

Despite this X and Y are **not independent**, as it is not true that

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

for all $(x,y) \in \mathbb{R}^2$. For example, on the region interior to the square circumscribing the unit circle, but exterior to the unit circle, $f_{X,Y}(x,y) = 0$, but $f_X(x) > 0$ and $f_Y(y) > 0$.

4 MARKS

3. (a) By using mgfs, we have that $U \sim N(0,2)$, as

$$M_U(t) = M_{Z_1}(t) M_{Z_2}(t) = e^{t^2/2} e^{t^2/2} = e^{\{t/\sqrt{2}\}^2}$$

4 MARKS

For V, from first principles

$$F_V(v) = P[V \leq v] = P[Z_1/Z_2 \leq v] = \int_{-\infty}^{\infty} \int_{-\infty}^{z_2 v} \phi(z_1) \phi(z_2) \, dz_1 dz_2$$

where ϕ is the standard normal pdf. Thus, differentiating wrt v under the integral, we have for $v \in \mathbb{R}$,

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty} z_2 \phi(z_2 v) \phi(z_2) \, dz_2 = \int_{-\infty}^{\infty} z_2 \frac{1}{2\pi} \exp\left\{ -\frac{1}{2}(z_2^2 v^2 + z_2^2) \right\} \, dz_2 \\ &= \frac{1}{2\pi} \left[-\frac{1}{1+v^2} \exp\left\{ -\frac{1}{2}(z_2^2 v^2 + z_2^2) \right\} \right]_{-\infty}^{\infty} = \frac{1}{\pi} \frac{1}{1+v^2} \end{aligned}$$

so $V \sim \mathit{Cauchy}$.

6 MARKS

An alternative method of proof uses the joint transformation theorem.

$$\left.\begin{array}{rcl} U &=& Z_1 + Z_2 \\ V &=& Z_1/Z_2 \end{array}\right\} \qquad \Longleftrightarrow \qquad \left\{\begin{array}{rcl} Z_1 &=& UV/(V+1) \\ Z_2 &=& U/(V+1) \end{array}\right.$$

so that the Jacobian is equal to

$$\left|\begin{array}{cc} \dfrac{\partial z_1}{\partial u} & \dfrac{\partial z_1}{\partial v} \\ & \\ \dfrac{\partial z_2}{\partial u} & \dfrac{\partial z_2}{\partial u} \end{array}\right| = \left|\begin{array}{cc} \dfrac{v}{v+1} & \dfrac{u}{(v+1)^2} \\ & \\ \dfrac{1}{v+1} & -\dfrac{u}{(v+1)^2} \end{array}\right| = \frac{|u|}{(v+1)^2}$$

and thus the joint pdf is

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(uv/(v+1), u/(v+1))|J(u,v)| \qquad (1) \\ &= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left[\frac{u^2v^2}{(v+1)^2} + \frac{u^2}{(v+1)^2}\right]\right\}\frac{|u|}{(v+1)^2} \qquad (2) \\ &= \frac{1}{2\pi}\exp\left\{-\frac{u^2}{2}\frac{v^2+1}{(v+1)^2}\right\}\frac{|u|}{(v+1)^2}. \qquad (3) \end{aligned}$$

Integrating out u yields

$$\begin{aligned} f_V(v) &= \int_{-\infty}^{\infty}\frac{1}{2\pi}\exp\left\{-\frac{u^2}{2}\frac{v^2+1}{(v+1)^2}\right\}\frac{|u|}{(v+1)^2}\,du \\ &= \frac{1}{\pi}\int_0^{\infty}\exp\left\{-\frac{u^2}{2}\frac{v^2+1}{(v+1)^2}\right\}\frac{u}{(v+1)^2}\,du \\ &= \frac{1}{\pi}\left[-\frac{1}{1+v^2}\exp\left\{-\frac{u^2}{2}\frac{v^2+1}{(v+1)^2}\right\}\right]_0^{\infty} \\ &= \frac{1}{\pi}\frac{1}{1+v^2} \qquad v\in\mathbb{R} \end{aligned}$$

From equation (3), we note that $f_{U,V}(u,v)$ **does not factorize** into a product of a function of u and a function of v, and thus U and V are **not independent**.

5 MARKS

(b) (i) From notes

$$f(x|\theta,\sigma) = \frac{1}{\sigma}f_V((x-\theta)/\sigma) = \frac{1}{\sigma\pi}\frac{1}{1+(x-\theta)^2/\sigma^2}$$

which is symmetric about θ as

$$(-(x-\theta))^2 = (x-\theta)^2$$

5 MARKS

(ii) The expectation of the *Cauchy* distribution is **not finite**, as

$$\int_{-\infty}^{\infty} x\frac{1}{\pi}\frac{1}{1+x^2}\,dx$$

does not converge. Hence the expectation of the distribution specified by $f(x|\theta,\sigma)$ is not finite either.

5 MARKS

4. (a) The pmf at issue is

$$f_X(x) = \binom{n+x-1}{x} \theta^n (1-\theta)^x \qquad x = 0, 1, 2, \ldots$$

where we treat θ as a single parameter, and n as a fixed constant in $\mathbb{Z}^+$, as in the case of the *Binomial* distribution.

(i) The pmf can be written as an exponential family distribution

$$f(x|\theta) = h(x)c(\theta)\exp\{w(\theta)t(x)\} \qquad x \in \mathbb{R}$$

where

$$h(x) = \binom{n+x-1}{x} I_{\{0,1,2,\ldots\}}(x) \quad c(\theta) = \theta^n \quad w(\theta) = \log(1-\theta) \quad t(x) = x$$

8 MARKS

(ii) The canonical parameter is

$$\eta = \log(1-\theta)$$

2 MARKS

(iii) From the formula sheet, the mgf is given by

$$\left(\frac{\theta}{1-e^t(1-\theta)}\right)^n \tag{4}$$

Now, consider $N = 1, 2, \ldots$. As

$$\left(\frac{\theta}{1-e^t(1-\theta)}\right)^n = \left\{\left(\frac{\theta}{1-e^t(1-\theta)}\right)^{n/N}\right\}^N = \{M(t)\}^N$$

it follows that the distribution is infinitely divisible if $M(t)$ is the mgf of a probability distribution, which **is** the case if

$$\binom{\alpha+x-1}{x} \theta^\alpha (1-\theta)^x \tag{5}$$

is a valid pmf when $\alpha = n/N$. But as

$$\sum_{x=0}^{\infty} \binom{\alpha+x-1}{x} (1-\theta)^x = \frac{1}{(1-(1-\theta))^\alpha} = \frac{1}{\theta^\alpha}$$

(the "negative binomial expansion"), it is the case that equation (5) is a valid pmf, and therefore the form in equation (4) is infinitely divisible.

6 MARKS

(b) (i) For the Weibull distribution, from the formula sheet,

$$h_X(x) = \frac{f_X(x)}{1-F_X(x)} = \frac{\alpha\beta x^{\alpha-1} e^{-\beta x^\alpha}}{e^{-\beta x^\alpha}} = \alpha\beta x^{\alpha-1} \qquad x > 0$$

4 MARKS

(ii) The Weibull distribution is **not** in the exponential family, unless $\alpha = 1$, as the term

$$\beta x^\alpha$$

cannot be written as a sum of terms of the form

$$\sum_{j=1}^{k} w_j(\alpha, \beta) t_j(x).$$

5 MARKS