MATH 556 - EXERCISES 5: SOLUTIONS

1. (a) This is not an Exponential Family distribution; the support is parameter dependent.

   (b) This is an EF distribution with $m = 1$:

$$f(x;\theta) = \frac{\mathbb{1}_{\{1,2,3,\ldots\}}(x)}{x} \frac{-1}{\log(1-\theta)} \exp\{x \log\theta\} = \exp\{c(\theta)T(x) - A(\theta)\}h(x)$$

- $h(x) = \dfrac{\mathbb{1}_{\{1,2,3,\ldots\}}(x)}{x}$
- $A(\theta) = \log(-\log(1-\theta))$
- $c(\theta) = \log(\theta)$
- $T(x) = x$

so the natural parameter is $\eta = \log(\theta)$.

2. (a) Suppose that $\eta_1, \eta_2 \in \mathcal{H}$ and $0 \leq t \leq 1$. Then

$$\begin{aligned}\int h(x)e^{(t\eta_1+(1-t)\eta_2)^\top T(x)}\,dx &= \int h(x)e^{(t\eta_1)^\top T(x)}e^{((1-t)\eta_2)^\top T(x)}\,dx \\ &\leq \left\{\int h(x)e^{(t\eta_1)^\top T(x)}dx\right\}\left\{\int h(x)e^{((1-t)\eta_2)^\top T(x)}\,dx\right\} \\ &\leq \left\{\int h(x)e^{\eta_1^\top T(x)}dx\right\}^t\left\{\int h(x)e^{\eta_2^\top T(x)}\,dx\right\}^{(1-t)} < \infty\end{aligned}$$

so $t\eta_1 + (1-t)\eta_2 \in \mathcal{H}$.

   (b) By inspection

$$\log\frac{f_X(x;\eta_1)}{f_X(x;\eta_2)} = (\eta_1 - \eta_2)T(x) - (K(\eta_1) - K(\eta_2))$$

Note that this ratio is zero for all x **if and only if** $\eta_1 = \eta_2$, unless $T(x)$ is a constant, t_0, say, for all x. In this latter case, we have that

$$K(\eta) = \log\left\{\int h(x)\exp\{\eta t_0\}\,dx\right\} = \eta t_0$$

in which case

$$\log\frac{f_X(x;\eta_1)}{f_X(x;\eta_2)} = (\eta_1 - \eta_2)t_0 - (\eta_1 t_0 - \eta_2 t_0) = 0$$

also, for any η_1 and η_2. Hence we can conclude that the EF model is *identifiable*

$$f_X(x;\eta_1) = f_X(x;\eta_2) \qquad \Longleftrightarrow \qquad \eta_1 = \eta_2$$

unless $T(X)$ has a degenerate distribution (for a value $\eta_0 \in \mathcal{H}$).