MATH 556 - EXERCISES 4: SOLUTIONS

1. By direct calculation, we have by the theorem of total probability for $y \geq 2$,

$$f_Y(y) = P_Y[Y = y] = \sum_{x_1=1}^{\infty} P_{X_1,X_2}[X_1 = x_1, X_2 = y - x_1] = \sum_{x_1=1}^{y-1} P_{X_1}[X_1 = x_1] P_{X_2}[X_2 = y - x_1]$$

by independence. Thus

$$\begin{aligned} f_Y(y) &= \sum_{x_1=1}^{y-1} (1-\theta_1)^{x_1-1}\theta_1(1-\theta_2)^{y-x_1-1}\theta_2 = \frac{\theta_1\theta_2(1-\theta_2)^y}{(1-\theta_1)(1-\theta_2)} \sum_{x_1=1}^{y-1} \left(\frac{1-\theta_1}{1-\theta_2}\right)^{x_1} \\ &= \frac{\theta_1\theta_2(1-\theta_2)^y}{(1-\theta_1)(1-\theta_2)} \left(\frac{1-\theta_1}{1-\theta_2}\right) \frac{1 - \left(\dfrac{1-\theta_1}{1-\theta_2}\right)^{y-1}}{1 - \left(\dfrac{1-\theta_1}{1-\theta_2}\right)} \\ &= \theta_1\theta_2(1-\theta_2)^{y-2} \frac{1 - \left(\dfrac{1-\theta_1}{1-\theta_2}\right)^{y-1}}{1 - \left(\dfrac{1-\theta_1}{1-\theta_2}\right)} = \frac{\theta_1\theta_2}{\theta_1 - \theta_2}\left[(1-\theta_2)^{y-1} - (1-\theta_1)^{y-1}\right] \end{aligned}$$

Alternately, using probability generating functions (pgfs), we have that

$$G_Y(t) = G_{X_1}(t)G_{X_2}(t) = \frac{\theta_1 t}{1 - t(1-\theta_1)} \frac{\theta_2 t}{1 - t(1-\theta_2)}$$

which, on expansion as a power series in t, yields the same summation as above.

2. By inspection, we have for $(x_1, x_2) \in \mathbb{R}^2$

$$f_{X_1,X_2}(x_1, x_2) = c \exp\left\{-|x_1|\right\} |x_1| \exp\left\{-\frac{x_1^2 x_2^2}{2}\right\} = f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1)$$

where

$$\begin{aligned} f_{X_1}(x_1) &= \begin{cases} \dfrac{1}{2}\, e^{-|x_1|} & x_1 \in \mathbb{R}\backslash\{0\} \\ 0 & x_1 = 0 \end{cases} \\ f_{X_2|X_1}(x_2|x_1) &= \begin{cases} \left(\dfrac{x_1^2}{2\pi}\right)^{1/2} \exp\left\{-\dfrac{x_1^2 x_2^2}{2}\right\} & x_2 \in \mathbb{R} \text{ if } x_1 \neq 0 \\ 0 & x_2 \in \mathbb{R} \text{ if } x_1 = 0 \end{cases} \end{aligned}$$

Thus $c = 1/\sqrt{8\pi}$.

3. We have $f_R(r) = 6r(1-r)$, for $0 < r < 1$, and hence

$$F_R(r) = r^2(3 - 2r) \quad 0 < r < 1$$

with the usual cdf behaviour outside of this range.

- Circumference: $X_1 = 2\pi R$, so $\mathbb{X}_1 = (0, 2\pi)$, and from first principles, for $x_1 \in \mathbb{X}_1$,

$$F_{X_1}(x_1) = P_{X_1}[\, X_1 \le x_1 \,] = P_R[\, 2\pi R \le x_1 \,] = P_R[\, R \le x_1/2\pi \,]$$
$$= F_R(x_1/2\pi) = \frac{3x_1^2}{4\pi^2} - \frac{2x_1^3}{8\pi^3}$$
$$\Longrightarrow f_{X_1}(x_1) = \frac{6x_1}{8\pi^3}(2\pi - x_1) \qquad 0 < x_1 < 2\pi$$

- Area: $X_2 = \pi R^2$, so $\mathbb{X}_2 = (0, \pi)$, and from first principles, for $x_2 \in \mathbb{X}_2$, recalling that f_R is only positive when $0 < x_2 < \pi$,

$$F_{X_2}(x_2) = P_{X_2}[\, X_2 \le x_2 \,] = P_R[\, \pi R^2 \le x_2 \,] = P_R[\, R \le \sqrt{x_2/\pi} \,]$$
$$= F_R(x_2/2\pi) = \frac{3x_2}{\pi} - 2\left\{\frac{x_2}{\pi}\right\}^{3/2}$$
$$\Longrightarrow f_{X_2}(x_2) = 3\pi^{-3/2}(\sqrt{\pi} - \sqrt{x_2}) \qquad 0 < x_2 < \pi.$$

Finally, for the joint distribution, we have that $X_2 = \pi R^2 = \pi(X_1/(2\pi))^2 = X_1^2/(4\pi)$ so the joint pdf is degenerate along the line $x_2 = x_1^2/(4\pi)$, that is

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1) = \mathbb{1}_{(0,2\pi)}(x_1)\frac{6x_1}{8\pi^3}(2\pi - x_1)\mathbb{1}_{\{x_1^2/(4\pi)\}}(x_2)$$

4. If $\mathbb{X}^{(2)} = (0,1) \times (0,1)$ is the (joint) range of vector random variable (X, Y). We have

$$f_{X,Y}(x, y) = cx(1 - y) \qquad 0 < x < 1,\ 0 < y < 1$$

so that $f_{X,Y}(x, y) = f_X(x) f_Y(y)$ and $\mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$, where $\mathbb{X}$ and $\mathbb{Y}$ are the supports of X and Y respectively, and

$$f_X(x) = c_1 x \qquad \text{and} \qquad f_Y(y) = c_2(1 - y) \tag{1}$$

for some constants satisfying $c_1 c_2 = c$. Hence, the two sufficient conditions for independence (that the joint pdf factorizes into a function of one variable and a function of the other, and the support is a Cartesian product) are satisfied in (1), and X and Y are independent.

Secondly, we must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dxdy = 1 \qquad \therefore \qquad c^{-1} = \int_0^1\int_0^1 x(1 - y)\, dxdy = 1$$

and as

$$\int_0^1\int_0^1 x(1 - y)\, dxdy = \left\{\int_0^1 x\, dx\right\}\left\{\int_0^1 (1 - y)\, dy\right\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

we have $c = 4$.

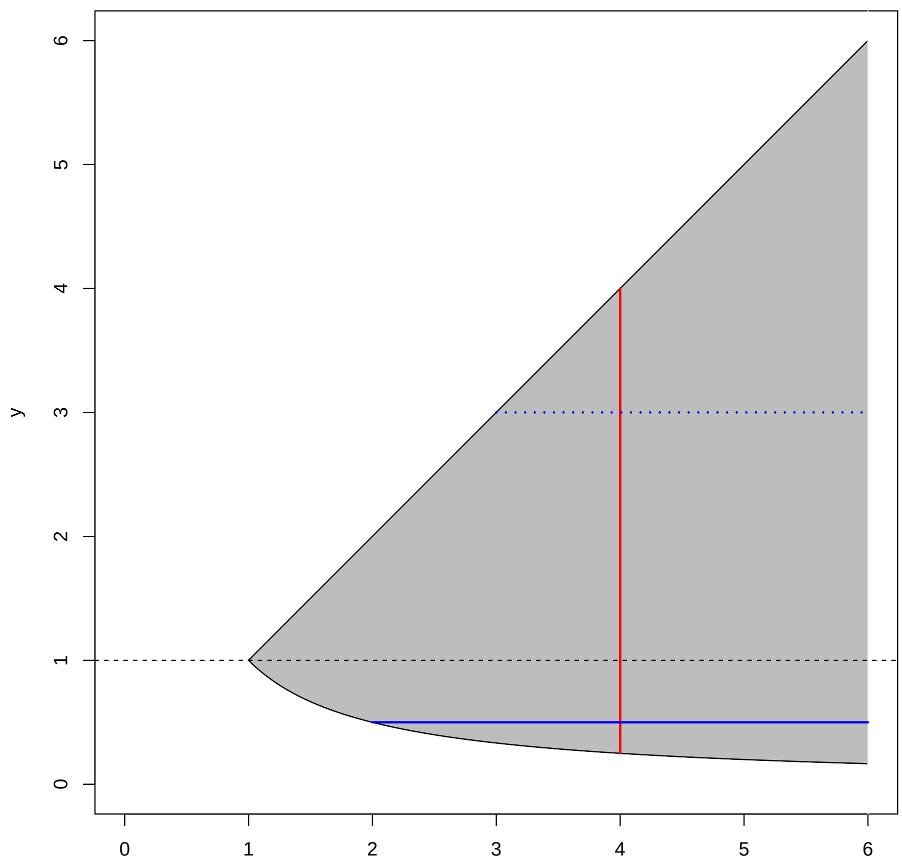
Finally, we have $A = \{(x, y) \,:\, 0 < x < y < 1\}$, and hence, recalling that the joint density is only non-zero when $x < y$, we first fix a y and integrate dx on the range $(0, y)$, and then integrate dy on the range $(0, 1)$, that is

$$P_{X,Y}[\, X < Y \,] = \iint_A f_{X,Y}(x, y)\, dxdy = \int_0^1\left\{\int_0^y 4x(1 - y)\, dx\right\} dy$$
$$= \int_0^1\left\{\int_0^y x\, dx\right\} 4(1 - y)\, dy = \int_0^1 2y^2(1 - y)\, dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^4\right]_0^1 = \frac{1}{6}$$

5. First sketch the support of the density; this will make it clear that the boundaries of the support are different for $0 < y \leq 1$ and $y > 1$.

   In this figure

   - the gray shaded region is the support of the joint pdf;
   - the solid red line indicates the range of integration dy for a fixed x; this range is always $y = 1/x$ to $y = x$;
   - the solid blue line indicates the range of integration dx for a fixed $y < 1$; this range is always $x = 1/x$ to $x = \infty$.
   - the dotted blue line indicates the range of integration dx for a fixed $y > 1$; this range is always $x = y$ to $x = \infty$.



   (i) The marginal distributions are given by

$$f_X(x) \quad = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{1/x}^{x} \frac{1}{2x^2} y\,dy = \frac{1}{2x^2}(\log x - \log(1/x)) = \frac{\log x}{x^2} \qquad 1 \leq x$$

$$f_Y(y) \quad = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \displaystyle\int_{1/y}^{\infty} \frac{1}{2x^2 y}\,dx = \frac{1}{2} & 0 \leq y \leq 1 \\ \displaystyle\int_{y}^{\infty} \frac{1}{2x^2 y}\,dx = \frac{1}{2y^2} & 1 \leq y \end{cases}$$

(ii) Conditionals:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{x^2 y} & 1/y \leq x \text{ if } 0 \leq y \leq 1 \\ \\ \dfrac{y}{x^2} & y \leq x \text{ if } 1 \leq y \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{2y \log x} \qquad 1/x \leq y \leq x \text{ if } x \geq 1$$

(iii) Marginal expectation of Y;

$$\mathbb{E}_Y[\,Y\,] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 \frac{y}{2}\,dy + \int_1^{\infty} \frac{1}{2y}\,dy = \infty$$

as the second integral is divergent.

6. (i) We set

$$\begin{array}{l} U = X/Y \\ V = -\log(XY) \end{array} \iff \begin{array}{l} X = U^{1/2}e^{-V/2} \\ Y = U^{-1/2}e^{-V/2} \end{array}$$

note that, as X and Y lie in $(0,1)$ we have $XY < X/Y$ and $XY < Y/X$, giving constraints $e^{-V} < U$ and $e^{-V} < 1/U$, so that $0 < e^{-V} < \min\{U, 1/U\}$. The Jacobian of the transformation is

$$|J(u,v)| = \left| \begin{array}{cc} \dfrac{u^{-1/2}e^{-v/2}}{2} & -\dfrac{u^{1/2}e^{-v/2}}{2} \\ \\ -\dfrac{u^{-3/2}e^{-v/2}}{2} & -\dfrac{u^{-1/2}e^{-v/2}}{2} \end{array} \right| = u^{-1}e^{-v}/2.$$

Hence

$$f_{U,V}(u,v) = u^{-1}e^{-v}/2 \qquad 0 < e^{-v} < \min\{u, 1/u\}\,,\ u > 0$$

The corresponding marginals are given below: let $g(y) = -\log(\min\{u, 1/u\})$, then

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,dv = \int_{g(y)}^{\infty} \frac{e^{-v}}{2u}\,dv = \left[-\frac{e^{-v}}{2u}\right]_{g(y)}^{\infty} = \frac{\min\{u, 1/u\}}{2u} \qquad u > 0$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\,du = \int_{e^{-v}}^{e^{v}} \frac{e^{-v}}{2u}\,du = \left[\frac{\log u}{2}e^{-v}\right]_{e^{-v}}^{e^{v}} = ve^{-v} \qquad v > 0$$

(ii) Now let

$$\begin{array}{l} V = X + Y \\ \\ Z = X - Y \end{array} \iff \begin{array}{l} X = \dfrac{V+Z}{2} \\ \\ Y = \dfrac{V-Z}{2} \end{array}$$

and the Jacobian of the transformation is $1/2$. The transformed variables take values on the square A in the (V, Z) plane with corners at $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$ bounded by the lines $z = -v$, $z = 2 - v$, $z = v$ and $z = v - 2$. Then

$$f_{V,Z}(v,z) = \frac{1}{2} \qquad (v,z) \in A$$

and zero otherwise (sketch the square A). Hence, integrating in horizontal strips in the (V, Z) plane,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{V,Z}(v,z)\, dv = \begin{cases} \displaystyle\int_{-z}^{2+z} \frac{1}{2}\, dv & = 1+z \qquad -1 < z \leq 0 \\ \displaystyle\int_{z}^{2-z} \frac{1}{2}\, dv & = 1-z \qquad 0 < z < 1 \end{cases}$$

7. (a) Random variable $\mathbb{1}_B(X)$ takes values on the set $\{0,1\}$, with

$$P_{\mathbb{1}_B(X)}[\mathbb{1}_B(X) = 1] = P_X[X \in B] = \theta_B$$

say, so $\mathbb{1}_B(X) \sim Bernoulli(\theta_B)$, with expectation, from the formula sheet, θ_B

(b) Let $\mathbb{1}_B(\mathbf{X})$ be the scalar indicator random variable associated with the event $\mathbf{X} \in B$. Then from above, we have that

$$\mathbb{E}_{\mathbb{1}_B(\mathbf{X})}[\mathbb{1}_B(\mathbf{X})] = P_{\mathbf{X}}[\mathbf{X} \in B]$$

which indicates that we can construct the approximation

$$\widehat{\mathbb{E}}_{\mathbb{1}_B(\mathbf{X})}[\mathbb{1}_B(\mathbf{X})] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_B(\mathbf{x}_i)$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_N$ are an independent sample from the specified Normal distribution. The following `R` code implements this:

```
library(MASS)
N<-10000
Sigma<-matrix(c(1,0.2,-0.5,0.2,2.0,-0.1,-0.5,-0.1,2.0),3,3,byrow=T)
set.seed(101)
for(irep in 1:5){
   X<-mvrnorm(N,mu=c(0,0,0),Sigma)
   IndX<-X[,1]+X[,3]-(X[,1]^2+X[,2]^2) > 0
   E<-sum(IndX)/N
   print(format(E,nsmall=4))
}
```

and yields the results

```
[1] 0.1513
[1] 0.1433
[1] 0.1475
[1] 0.1529
[1] 0.1484
```

By using a very large N, we can discover that the true value is 0.1468 to four decimal places.