Class Notes for MATH 366.

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Complex Number Basics

Complex numbers take the form $z = a + ib$ where a and b are real. The symbol i is precisely that — a symbol. The set of all complex numbers is denoted $\mathbb{C}$. We call a the real part of z and use the notation $a = \Re z$ and b the imaginary part of z with the notation $b = \Im z$. If $\Im z = b = 0$, then we say that z is real and identify it to the real number a. Thus we view the real line $\mathbb{R}$ as a subset of $\mathbb{C}$.

We define addition and multiplication according to the following laws

$$\begin{aligned}
(a_1 + ib_1) + (a_2 + ib_2) &= (a_1 + a_2) + i(b_1 + b_2), \\
(a_1 + ib_1) \cdot (a_2 + ib_2) &= (a_1 b_1 - a_2 b_2) + i(a_1 b_2 + a_2 b_1).
\end{aligned}$$

It can be shown that addition is commutative and associative. The multiplication is also commutative and associative and the standard distributive laws hold.

A particular case of multiplication is multiplication by a real number

$$t \cdot (a_2 + ib_2) = (ta_2 + itb_2)$$

from putting $a_1 = t$, $b_1 = 0$ in the multiplication law and now it comes that

$$t_1 \cdot (a_1 + ib_1) + t_2 \cdot (a_2 + ib_2) = (t_1 a_1 + t_2 a_2) + i(t_1 b_1 + t_2 b_2)$$

and it follows that we can view $\mathbb{C}$ with addition and real multiplication as a two-dimensional real vector space. Hence the expression ***the complex plane*** . So, in this analogy, we are indentifying

$$(a + ib) \in \mathbb{C} \longleftrightarrow (a, b) \in \mathbb{R}^2$$

Following the analogy, complex addition is just vector addition in the plane and multiplication by a real number is scalar multiplication in the plane.

The multiplication law is based on the idea that $i^2 = -1$. Sometimes we hear it said that i is the square root of -1. The key point being that there is no *real* square root of -1. A more sophisticated approach to complex numbers would define $\mathbb{C} = \mathbb{R}[i]/I$ the quotient of the ring of polynomials in the indeterminate i with real coefficients by the ideal consisting of all such polynomials that are multiples of $i^2 + 1$. However, as far as we are concerned this is overkill[1]! But in any case, the complex numbers form a commutative ring with identity, the identity being $1 + i0$ which we will simply denoted by 1.

In fact $\mathbb{C}$ is a field. This means that if $a + ib \neq 0$ (translate logically to $(a, b) \neq (0, 0)$, or equivalently not both $a = 0$ and $b = 0$), we have

$$(a + ib) \cdot (x + iy) = (x + iy) \cdot (a + ib) = 1$$

where $x = a(a^2 + b^2)^{-1}$ and $y = -b(a^2 + b^2)^{-1}$. These are both well defined since $a^2 + b^2 > 0$ since not both $a = 0$ and $b = 0$. Thus every non-zero element of $\mathbb{C}$ possesses a reciprocal (multiplicative inverse) in $\mathbb{C}$.

Some additional definitions are as follows. The ***complex conjugate*** of a complex number $z = a+ib$ is $a-ib$ which actually means $a+i(-b)$ always assuming that a and b are real. It is denoted by $\overline{z}$. Visually, the mapping $z \mapsto \overline{z}$ takes a point in the complex plane to its reflection in the x-axis. It is a real linear mapping i.e. we have

$$\overline{t_1 z_1 + t_2 z_2} = t_1 \overline{z_1} + t_2 \overline{z_2}$$

for t_1 and t_2 real and furthermore a ring homomorphism i.e. $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$. Also it is involutary, namely $\overline{\overline{z}} = z$.

For a and b real, we define $|a + ib| = \sqrt{a^2 + b^2}$, the Euclidean norm of (a, b) in $\mathbb{R}^2$. This called the ***modulus*** or ***absolute value*** of the complex number $a + ib$. It is both a norm, in particular sublinear over $\mathbb{R}$

$$|t_1 z_1 + t_2 z_2| \leq |t_1||z_1| + |t_2||z_2|, \qquad t_1, t_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{C}$$

and satisfies $|z| = 0 \Rightarrow z = 0$ as well as being multiplicative

$$|z_1 z_2| = |z_1||z_2|.$$

There are many other important identities and inequalities which we list below.

[1] We could also say that $\mathbb{C}$ is the *splitting field* of the polynomial $i^2 + 1$ over $\mathbb{R}$, but this would be beyond overkill!

$z\overline{z} = \overline{z}z = \|z\|^2$	$\frac{1}{z} = \frac{\overline{z}}{\|z\|^2}$
$\Re z = \frac{z+\overline{z}}{2}$	$\Im z = \frac{z-\overline{z}}{2i}$
$\|\overline{z}\| = \|z\|$	$\|z^{-1}\| = \|z\|^{-1}$
$\|z+w\|^2 = \|z\|^2 + 2\Re\overline{z}w + \|w\|^2$	$\|z-w\|^2 = \|z\|^2 - 2\Re\overline{z}w + \|w\|^2$
$\|z_1 - z_3\| \leq \|z_1 - z_2\| + \|z_2 - z_3\|$	$\|z \pm w\| \leq \|z\| + \|w\|$
$\|\Re z\| \leq \|z\|$	$\|\Im z\| \leq \|z\|$

1.1 Polar Representation

The polar representation of complex numbers corresponds with the representation of points in the plane in polar coordinates. We write

$$z = r\cos(\theta) + ir\sin(\theta)$$

and it is clear that $r = \sqrt{x^2+y^2} = |z|$. The quantity θ is only defined if $z \neq 0$ and then only up to an integer multiple of 2π, that is if θ is a possible value, then so is $\theta + 2n\pi$ where $n \in \mathbb{Z}$. The quantity θ is called the ***argument*** of z. It is a consequence of the addition laws for cos and sin that

$$\begin{aligned}
&(r_1\cos(\theta_1) + ir_1\sin(\theta_1))(r_2\cos(\theta_2) + ir_2\sin(\theta_2)) \\
&= r_1r_2\Big(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\Big) \\
&\quad + ir_1r_2\Big(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)\Big) \\
&= r_1r_2\cos(\theta_1+\theta_2) + ir_1r_2\sin(\theta_1+\theta_2)
\end{aligned}$$

It follows that to take the product of two complex numbers, the modulus of the product is the product of the moduli, but the argument of the product is the sum of the arguments.

To cut down on writing we sometimes see $\operatorname{cis}(\theta) = \cos(\theta) + i\sin(\theta)$ and then we can write a complex number in polar representation as $z = r\operatorname{cis}(\theta)$. In fact, we will see later that $\operatorname{cis}(\theta) = \exp(i\theta)$ where the exponential function of a complex argument is defined by its power series. We have

$$\operatorname{cis}(\theta_1 + \theta_2) = \operatorname{cis}(\theta_1)\operatorname{cis}(\theta_2)$$

and also

$$\operatorname{cis}(n\theta) = \operatorname{cis}(\theta)^n$$

for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and $n \in \mathbb{Z}$. Other useful relations are

$$\overline{\operatorname{cis}(\theta)} = \operatorname{cis}(-\theta), \quad |\operatorname{cis}(\theta)| = 1.$$

These identities make it possible to solve easily equations of the type $z^n = \zeta$ with ζ given and unknown z. Let $z = r\operatorname{cis}(\theta)$ and $\zeta = \rho\operatorname{cis}(\phi)$ be the corresponding polar representations. Then we get $r^n = \rho$ and $n\theta = \phi \pmod{2\pi}$. Thus $r = \rho^{\frac{1}{n}}$ and $\theta = \dfrac{\phi + 2k\pi}{n}$ where k is an integer. But the latter sequence is periodic with period n taking account of the equivalence of arguments that differ by an integer multiple of 2π. Thus the solutions are given by

$$\theta = \frac{\phi + 2k\pi}{n}, \quad k = 0, 1, 2, \ldots, n-1$$

In particular the n^{th} roots of unity — the solutions of the equation $z^n = 1$ are given by $z = \operatorname{cis}(2k\pi/n)$ for $k = 0, 1, 2, \ldots, n-1$.

Since polynomials play a key role in complex analysis, it's worth observing that a polynomial of degree n can have no more then n roots. Let

$$p(z) = z^n + p_1 z^{n-1} + p_2 z^{n-2} + \cdots + p_{n-1} z + p_n$$

and suppose that ζ is a root, i.e. $p(\zeta) = 0$. Then

$$p(z) = p(z) - p(\zeta) = \sum_{k=0}^{n} p_{n-k}(z^k - \zeta^k).$$

But, for each nonnegative integer k, $(z^k - \zeta^k)$ is divisible by $z - \zeta$ since

$$z^k - \zeta^k = (z - \zeta)(z^{k-1} + \zeta z^{k-2} + \cdots + \zeta^{k-2} z + \zeta^{k-1})$$

and it follows that $p(z)$ is divisible by $z - \zeta$. The quotient of this division is a polynomial of degree $n - 1$ and a simple induction argument finishes the proof.

We will see later in the course that every monic[2] polynomial of degree n with complex coefficients can be factored as

$$p(z) = \prod_{k=1}^{n} (z - \zeta_k)$$

for some $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$.

Finally, let us not lose sight of the fact that $\mathbb{C}$ is a normed vector space over $\mathbb{R}$ and hence a metric space with distance function $d(z_1, z_2) = |z_1 - z_2|$. As a metric space, it inherits all the baggage that goes with metric spaces — open subsets, closed subsets, convergent sequences and, where mappings are defined, continuity. In particular, since $\mathbb{C}$ is finite dimensional over $\mathbb{R}$ we see that $\mathbb{C}$ is complete as a metric space. Furthermore, according to the Heine–Borel Theorem, every closed bounded subset of $\mathbb{C}$ is compact. Actually, in this course we will not use the open covering version of compactness. We will work with sequential compactness.

1.2 Complex Multiplication as a Real Linear Mapping

Consider multiplication by the complex number $a + ib$. We have

$$(u + iv) = (a + ib) \cdot (x + iy) = (ax - by) + i(bx + ay)$$

or $u = ax - by$ and $v = bx + ay$ which can be viewed as a real linear mapping from $\mathbb{R}^2$ to itself

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We therefore obtain the following observation. A real linear mapping from $\mathbb{R}^2$ to itself can be realised as a complex multiplication if and only if its matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

satisfies $a_{11} = a_{22}$ and $a_{21} = -a_{12}$.

[2] A polynomial is *monic* iff its leading coefficient is unity.

2

Analytic Functions of a Complex Variable

Let us consider a function defined by a power series expansion with complex coefficients

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \tag{2.1}$$

The following theorem tells us where this series converges.

THEOREM 1 *There is a "number" $\rho \in [0, \infty]$ such that the series (2.1) converges if $|z| < \rho$ and does not converge if $|z| > \rho$.*

There are two extreme cases. In the case $\rho = 0$, the series converges only if $z = 0$. In this case, the series is for all intents and purposes useless. The other extreme case is when $\rho = \infty$ and then the series converges for all complex z. We give two proofs, the first a simple-minded one and the second uses the root test and actually produces a formula for ρ. The number ρ is called the ***radius of convergence*** .

First proof.

The series always converges for $z = 0$, so let us define

$$\rho = \sup\{|z|; \text{Series (2.1) converges}\},$$

with the understanding that $\rho = \infty$ in case the set is unbounded above. Then, by definition, $|z| > \rho$ implies that (2.1) does not converge. It remains to show that $|z| < \rho$ implies that (2.1) converges. In this case, there exists ζ with $|z| < |\zeta|$ such that $a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \cdots$ converges, for otherwise $|z|$ would be an upper bound for the set over which the sup was taken. Therefore $a_n\zeta^n \longrightarrow 0$ as

$n \longrightarrow \infty$. So, there is a constant C such that $|a_n \zeta^n| \le C$ for all $n \in \mathbb{Z}^+$. But, now $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely by comparison with a geometric series

$$\sum_{n=0}^{\infty} |a_n z^n| \le \sum_{n=1}^{\infty} |a_n \zeta^n| \left| \frac{z}{\zeta} \right|^n \le \sum_{n=0}^{\infty} C \left| \frac{z}{\zeta} \right|^n < \infty$$

■

Note that the proof actually shows that (2.1) converges *absolutely* for $|z| < \rho$. Furthermore the proof shows that if $0 < r < \rho$, then the series converges uniformly on the set $|z| \le r$. Another way of saying this is that the series converges uniformly on the (sequentially) compact subsets of $\{z; z \in \mathbb{C}, |z| < \rho\}$. A further consequence is that the function defined by a power series is continuous in $\{z; z \in \mathbb{C}, |z| < \rho\}$.

Second proof. By the root test, series (2.1) converges if $\limsup_{n\to\infty} |a_n z^n|^{\frac{1}{n}} < 1$ and does not converge if $\limsup_{n\to\infty} |a_n z^n|^{\frac{1}{n}} > 1$, because the *proof* of the root test shows that the terms do not tend to zero. This gives the formula

$$\rho = \liminf_{n\to\infty} |a_n|^{-\frac{1}{n}}$$

which has to be interpreted by taking $|a_n|^{-\frac{1}{n}} = \infty$ if $a_n = 0$ and $\rho = \infty$ if $a_n = 0$ eventually, or if $\inf_{n \ge N} |a_n|^{-\frac{1}{n}}$ tends properly to ∞ as $N \longrightarrow \infty$.

So, in the complex case, we get a disk of convergence with convergence holding on the open disk and failing on the interior of its complement. In general, not much can be said about convergence on the boundary of the disk $|z| = \rho$.

Power series allow us to define many of the elementary functions in the complex setting.

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos(z) = \sum_{n=0}^{\infty} (-)^n \frac{z^{2n}}{(2n)!}$$

$$\sin(z) = \sum_{n=0}^{\infty} (-)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

It is easy to check that all the power series listed above have an infinite radius of convergence and therefore define functions in the entire complex plane.

2.1 Analytic Functions

We make the following definition.

DEFINITION *Let Ω be a nonempty open subset of $\mathbb{C}$. Let $f : \Omega \longrightarrow \mathbb{C}$. Then f is **analytic** if and only if for each point $\alpha \in \Omega$, there is a power series expansion*

$$\sum_{n=0}^{\infty} a_n(\alpha)(z-\alpha)^n \tag{2.2}$$

with strictly positive radius $\rho(\alpha)$ such that the sum of the series 2.2 equals $f(z)$ in the disk $|z-\alpha| < \rho(\alpha)$.

These functions are the main subject matter of this course. We will see later that the hypotheses are too strong. It will in fact suffice for f to possess a complex derivative at every point z of Ω in order for f to be analytic in Ω.

There is also a concept of analyticity on the real line that is sometimes used.

DEFINITION *Let U be a nonempty open subset of $\mathbb{R}$. Let $f : \Omega \longrightarrow \mathbb{R}$ or $\mathbb{C}$. Then f is **real analytic** if and only if for each point $\alpha \in U$, there is a power series expansion*

$$\sum_{n=0}^{\infty} a_n(\alpha)(x-\alpha)^n \tag{2.3}$$

with strictly positive radius $\rho(\alpha)$ such that the sum of the series 2.3 equals $f(x)$ in the interval $\alpha - \rho(\alpha) < x < \alpha + \rho(\alpha)$.

Of course, if f is real-valued, then the coefficients $a_n(\alpha)$ will be real.

Since the series 2.3 will converge in the disk $\{x; x \in \mathbb{C}, |x-\alpha| < \rho(\alpha)\}$, we may let

$$\Omega = \bigcup_{\alpha \in U} \{z; z \in \mathbb{C}, |z-\alpha| < \rho(\alpha)\}$$

and suspect that f will have an extension $\tilde{f}$ to Ω. This is in fact the case, but a little beyond our scope at this point in the course.

2.2 The Complex Exponential

LEMMA 2 *For $z_1, z_2 \in \mathbb{C}$ we have*

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$$

Proof. Let $\exp_N(z) = \displaystyle\sum_{n=0}^{N} \frac{z^n}{n!}$, then we get

$$\exp_N(z_1)\exp_N(z_2) = \sum_{n_1=0}^{N}\sum_{n_2=0}^{N} \frac{z_1^{n_1}}{n_1!}\frac{z_2^{n_2}}{n_2!}.$$

On the other hand

$$\exp_N(z_1+z_2) = \sum_{n=0}^{N} \frac{(z_1+z_2)^n}{n!} = \sum_{n=0}^{N} \frac{1}{n!}\, {}^nC_{n_1}\, z_1^{n_1} z_2^{n-n_1} = \sum_{\substack{n_1,n_2 \geq 0 \\ n_1+n_2 \leq N}} \frac{z_1^{n_1}}{n_1!}\frac{z_2^{n_2}}{n_2!}$$

It follows that

$$\exp_N(z_1)\exp_N(z_2) - \exp_N(z_1+z_2) = \sum_{\substack{0 \leq n_1,n_2 \leq N \\ n_1+n_2 > N}} \frac{z_1^{n_1}}{n_1!}\frac{z_2^{n_2}}{n_2!}$$

and at this point we put in the absolute values

$$|\exp_N(z_1)\exp_N(z_2) - \exp_N(z_1+z_2)| \leq \sum_{\substack{0 \leq n_1,n_2 \leq N \\ n_1+n_2 > N}} \frac{|z_1|^{n_1}}{n_1!}\frac{|z_2|^{n_2}}{n_2!}.$$

Now if $n_1 + n_2 > N$ then either $n_1 > \frac{1}{2}N$ or $n_2 > \frac{1}{2}N$ (or both). Hence we have

$$\sum_{\substack{0 \leq n_1, n_2 \leq N \\ n_1 + n_2 > N}} \frac{|z_1|^{n_1}}{n_1!} \frac{|z_2|^{n_2}}{n_2!} \leq \sum_{n_1 > \frac{1}{2}N} \sum_{n_2=0}^{N} \frac{|z_1|^{n_1}}{n_1!} \frac{|z_2|^{n_2}}{n_2!} + \sum_{n_2 > \frac{1}{2}N} \sum_{n_1=0}^{N} \frac{|z_1|^{n_1}}{n_1!} \frac{|z_2|^{n_2}}{n_2!}$$
$$\leq \exp(|z_2|) \sum_{n_1 > \frac{1}{2}N} \frac{|z_1|^{n_1}}{n_1!} + \exp(|z_1|) \sum_{n_2 > \frac{1}{2}N} \frac{|z_2|^{n_2}}{n_2!}$$

The expression $\displaystyle\sum_{n > \frac{1}{2}N} \frac{|z|^n}{n!}$ is a tail sum of the (convergent) series for $\exp(|z|)$ and hence tends to zero as N tends to infinity. It follows that $\exp_N(z_1)\exp_N(z_2) - \exp_N(z_1 + z_2) \longrightarrow 0$ as $N \to \infty$. But $\exp_N(z_j)$ converges to $\exp(z_j)$ for $j = 1, 2$ and $\exp_N(z_1 + z_2)$ converges to $\exp(z_1 + z_2)$. it follows that $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ as required. ■

It is easy to check from the power series definitions that if y is real, then

$$\exp(iy) = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \cos(y) + i\sin(y)$$

because $i^{2k} = (-1)^k$ and $i^{2k+1} = (-1)^k i$. So, if x is also real, we have

$$\exp(x + iy) = \exp(x)\exp(iy) = e^x \cos(y) + ie^x \sin(y)$$

allowing us to understand for an arbitrary complex number z, what the real and imaginary parts of $\exp(z)$ are in terms of the real and imaginary parts of z.

2.3 Manipulation of Power Series

Power series can be manipulated in obvious ways. The standard theorems concerning linear combinations, products, quotients and compositions hold good in the complex case. The proofs are essentially the same as those given in MATH 255. For the sake of the record, we repeat those proofs here, but we will eventually establish the results by other means.

In this section we will assume that

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \tag{2.4}$$

with radius r and for $|z| < r$ and that

$$g(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots$$

with radius s and for $|z| < s$. We already know from general principles the following result.

PROPOSITION 3 *The series* $\sum_{n=0}^{\infty}(\lambda a_n + \mu b_n)z^n$ *has radius at least* $\min(r, s)$ *and it converges to* $\lambda f(z) + \mu g(z)$ *for* $|z| < \min(r, s)$.

It is easy to find examples where the radius of $\sum_{n=0}^{\infty}(\lambda a_n + \mu b_n)z^n$ is strictly larger than $\min(r, s)$.

THEOREM 4 *The formal product series* $\sum_{n=0}^{\infty} c_n z^n$ *has radius of convergence at least* $\min(r, s)$ *and it converges to* $f(z)g(z)$ *for* $|z| < \min(r, s)$. *Explicit formulæ for* c_n *are given by*

$$c_n = \sum_{p=0}^{n} a_p b_{n-p} = \sum_{q=0}^{n} a_{n-q} b_q,$$

and furthermore

$$\sum_{n=0}^{\infty} |c_n| t^n \leq \left\{ \sum_{p=0}^{\infty} |a_p| t^p \right\} \left\{ \sum_{q=0}^{\infty} |b_q| t^q \right\} \tag{2.5}$$

for $0 \leq t < \min(r, s)$.

Proof. Let $0 \leq |z| \leq t < \min(r, s)$ and $\epsilon > 0$. We denote by

$$f_N(z) = \sum_{p=0}^{N} a_p z^p$$

$$g_N(z) = \sum_{q=0}^{N} b_q z^q$$

$$h_N(z) = \sum_{n=0}^{N} c_n z^n$$

then, a tricky calculation shows that

$$f_N(z)g_N(z) - h_N(z) = \sum_{0\le p,q\le N} a_p b_q z^{p+q} - \sum_{\substack{0\le p,q\\ p+q\le N}} a_p b_q z^{p+q} = \sum_{\substack{0\le p,q\le N\\ p+q>N}} a_p b_q z^{p+q}.$$

This gives

$$|f_N(z)g_N(z) - h_N(z)| \le \sum_{\substack{0\le p,q\le N\\ p+q>N}} |a_p||b_q|t^{p+q}$$

and, since $p+q > N$ implies $p > N/2$ or $q > N/2$

$$\begin{aligned}
&\le \sum_{\substack{0\le p\le N\\ N/2<q\le N}} |a_p||b_q|t^{p+q} + \sum_{\substack{0\le q\le N\\ N/2<p\le N}} |a_p||b_q|t^{p+q}\\
&\le \left(\sum_{p=0}^{N} |a_p|t^p\right) \sum_{q=\lceil\frac{N}{2}\rceil}^{N} |b_q|t^q + \left(\sum_{q=0}^{N} |b_q|t^q\right) \sum_{p=\lceil\frac{N}{2}\rceil}^{N} |a_p|t^p\\
&\le \left(\sum_{p=0}^{\infty} |a_p|t^p\right) \sum_{q=\lceil\frac{N}{2}\rceil}^{\infty} |b_q|t^q + \left(\sum_{q=0}^{\infty} |b_q|t^q\right) \sum_{p=\lceil\frac{N}{2}\rceil}^{\infty} |a_p|t^p\\
&< \epsilon
\end{aligned}$$

if N is large enough. But, on the other hand, we also have

$$|f(z)g(z) - f_N(z)g_N(z)| < \epsilon$$

if N is large enough. Therefore, for N large enough

$$|f(z)g(z) - h_N(z)| < 2\epsilon.$$

Since ϵ is an arbitrary positive number this shows that the partial sums (h_N) of the formal product series converge to the product of the sums of the given series. This holds for all z with $|z| < t$, but since t may approach $\min(r,s)$, it holds for all z with $|z| < \min(r,s)$. So the radius of convergence of the formal product series is at least $\min(r,s)$.

To show (2.5) we remark that

$$\sum_{n=0}^{N} |c_n|t^n \le \sum_{\substack{0\le p,q\\ p+q\le N}} |a_p||b_q|t^{p+q} \le \sum_{0\le p,q\le N} |a_p||b_q|t^{p+q} = \left\{\sum_{p=0}^{N} |a_p|t^p\right\}\left\{\sum_{q=0}^{N} |b_q|t^q\right\}$$

and let N tend to ∞. ∎

EXAMPLE The radius of convergence of a product series can exceed the minimum of the individual radii. To see this, take

$$\frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + \cdots$$

and

$$\frac{1-z}{1+z} = 1 - 2z + 2z^2 - 2z^3 + 2z^4 + \cdots$$

both of which have radius 1. But, not surprisingly, the product series is

$$1 = 1 + 0z + 0z^2 + 0z^3 + 0z^4 + \cdots$$

which has infinite radius. □

COROLLARY 5 *Let $K \in \mathbb{Z}^+$. The formal K-fold product series $\displaystyle\sum_{n=0}^{\infty} c_{K,n} z^n$ of $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ has radius at least r and converges to $\Big(f(z)\Big)^K$. Furthermore*

$$\sum_{n=0}^{\infty} |c_{K,n}| t^n \le \left\{ \sum_{n=0}^{\infty} |a_n| t^n \right\}^K$$

for $0 \le t < r$.

EXAMPLE Again, the product series may have larger radius of convergence than the original. Consider

$$(1+2z)^{\frac{1}{2}} = 1 + z - \frac{1}{2!} z^2 + \frac{1 \cdot 3}{3!} z^3 - \frac{1 \cdot 3 \cdot 5}{4!} z^4 + \cdots$$

which has radius $\frac{1}{2}$. However, the square of this series is just $1 + 2z$ which has infinite radius. □

Now we come to compositions. Usually, the formal composition does not make sense. The formal K-fold product series has constant term $C_{K,0} = a_0^K$ and so the constant term of $g \circ f$ would be $\sum_{K=0}^{\infty} b_K a_0^K$ which is an infinite sum. So, in general, composition of power series is not a formal operation. However, if we suppose that $a_0 = 0$, then it does become a formal operation. In this special case, the series $\{\sum_{n=0}^{\infty} a_n z^n\}^K$ starts with the term in z^K (or later if $a_1 = 0$). Hence, in computing the coefficient of z^n in $g \circ f$ we need only consider $K = 0, 1, \ldots, n$. So each coefficient of $g \circ f$ is in fact a finite sum.

In practice, it is convenient to break up the discussion of general compositions of power series into two separate operations. One of these is the special type of composition with $a_0 = 0$ which is a formal operation and the other is the recentering of power series which is not a formal operation. We will deal with recentering later.

THEOREM 6 *Suppose that $a_0 = 0$. Then the formally composed series of $g \circ f$ has strictly positive radius and converges to $g(f(z))$ for z in some neighbourhood of 0. In most situations, one can say nothing about the radius of convergence, except that it is strictly positive. However, if s is infinite, then the formally composed series has radius of convergence at least r.*

Proof. Let

$$\varphi(t) = \sum_{n=1}^{\infty} |a_n| t^n.$$

The series has radius $r > 0$ and so φ is continuous at 0. Hence, there is a number $\rho > 0$ such that

$$0 \le t < \rho \qquad \Longrightarrow \qquad |\varphi(t)| < s.$$

Now we have for $|z| < \rho$, $|f(z)| = |\sum_{n=1}^{\infty} a_n z^n| \le \varphi(|z|) < s$, so that

$$g \circ f(z) = \sum_{K=0}^{\infty} b_K \Big(f(z)\Big)^K.$$

Note that if $s = \infty$, then we may take $\rho = r$ if r is finite or ρ to be any positive number (as large as we please) if $r = \infty$. Now, using the Corollary 5 we get

$$g \circ f(z) = \sum_{K=0}^{\infty} b_K \sum_{n=K}^{\infty} c_{K,n} z^n. \tag{2.6}$$

The inner sum could be taken from $n = 0$ to infinity, but $c_{K,n} = 0$ for $0 \le n < K$. What we would like to do is to interchange the order of summation in (2.6). This would yield

$$g \circ f(z) = \sum_{n=0}^{\infty} \left\{ \sum_{K=0}^{n} b_K c_{K,n} \right\} z^n.$$

and indeed, $\sum_{K=0}^{n} b_K c_{K,n}$ is the coefficient of z^n in the formal powers series for $g \circ f$. To justify this interchange, we must apply the theorem dealing with changing the order of summation (Fubini's Theorem). We need to show

$$\sum_{K=0}^{\infty} |b_K| \sum_{n=K}^{\infty} |c_{K,n}| t^n < \infty. \tag{2.7}$$

But, according to Corollary 5,

$$\sum_{n=K}^{\infty} |c_{K,n}| t^n \leq \Big(\varphi(t)\Big)^K$$

and (2.7) holds since $\varphi(t) < s$. The radius of convergence of the formally composed series is then at least ρ. In the special case $s = \infty$, (2.7) holds for $\rho = r$ if r is finite, or for every finite $\rho > 0$ if r is infinite. The radius of convergence of the series is therefore at least r. ∎

COROLLARY 7 *Suppose that $a_0 \neq 0$. Then*

$$\frac{1}{f(z)} = d_0 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots$$

with strictly positive radius. In fact, the coefficients $d_0, d_1, \ldots$ can be obtained by successively solving the recurrence relations

$$\begin{aligned}
1 &= a_0 d_0 \\
0 &= a_1 d_0 + a_0 d_1 \\
0 &= a_2 d_0 + a_1 d_1 + a_0 d_2 \\
0 &= a_3 d_0 + a_2 d_1 + a_1 d_2 + a_0 d_3
\end{aligned}$$

et cetera.

Proof. We can assume without loss of generality that $a_0 = 1$. Now, let us define $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(w) = (1 + w)^{-1}$. Then, applying Theorem 6, we see that

$$\frac{1}{f(z)} = \frac{1}{1 + h(z)} = (g \circ h)(z)$$

has a power series expansion with strictly positive radius. Once we know this, then both f and $\frac{1}{f}$ have expansions with strictly positive radius and

$$f(z) \cdot \frac{1}{f(z)} = 1$$

so the Product Theorem 2.5 allows us to conclude that the coefficients are in fact obtained by formal multiplication, leading to the recurrence relations cited above. ■

Finally we deal with recentering power series. This is not a formal power series operation. We will start with a power series centered at 0, namely

$$a_0 + a_1 z + a_2 z^2 + \cdots$$

Let us suppose that it has radius $r > 0$. Now let $|\alpha| < r$. Then we wish to expand the same gadget about $z = \alpha$

$$b_0 + b_1(z-\alpha) + b_2(z-\alpha)^2 + b_3(z-\alpha)^3 + \cdots \tag{2.8}$$

Substituting $w = z - \alpha$ and comparing the coefficient of w^n in

$$\sum_{k=0}^{\infty} a_k (w+\alpha)^k = \sum_{n=0}^{\infty} b_n w^n \tag{2.9}$$

we find the formula

$$b_n = \sum_{k=n}^{\infty} {}^k C_n \, a_k \alpha^{k-n}. \tag{2.10}$$

So the coefficients of the recentered series are infinite sums (as opposed to finite sums) and this is why the recentering operation is not an operation on formal power series.

THEOREM 8 *Under the hypotheses given above, the series* (2.10) *defining* b_n *converges for all* $n \in \mathbb{Z}^+$. *The radius of convergence of* (2.8) *is at least* $r - |\alpha|$. *Finally, the identity* (2.9) *holds provided that* $|w| < r - |\alpha|$.

Proof. Let $|\alpha| < \rho < r$ and $\sigma = \rho - |\alpha|$. Then

$$\begin{aligned}\sum_{n=0}^{\infty} \sigma^n \sum_{k=n}^{\infty} {}^kC_n \, |a_k||\alpha|^{k-n} &= \sum_{k=0}^{\infty} |a_k| \sum_{n=0}^{k} {}^kC_n \, \sigma^n |\alpha|^{k-n} \\ &= \sum_{k=0}^{\infty} |a_k|(\sigma + |\alpha|)^k < \infty\end{aligned}$$

since the order of summation can be interchanged for series of positive terms and since $\sigma + |\alpha| = \rho < r$. In particular it follows that for each fixed n, the inner series $\sum_{k=n}^{\infty} {}^kC_n \, |a_k||\alpha|^{k-n}$ converges and hence the series (2.10) converges absolutely for each $n \in \mathbb{Z}^+$. The same argument now shows that

$$\sum_{n=0}^{\infty} |b_n|\sigma^n \leq \sum_{n=0}^{\infty} \sigma^n \sum_{k=n}^{\infty} {}^kC_n \, |a_k||\alpha|^{k-n} < \infty$$

and so (2.8) converges absolutely whenever $|w| < r - |\alpha|$. So the radius of convergence of the recentered series is at least $r - |\alpha|$. Finally, we use Fubini's theorem to show that (2.9) holds. Effectively, since

$$\sum_{k=0}^{\infty} |a_k| \sum_{n=0}^{k} {}^kC_n \, |w|^n |\alpha|^{k-n} < \infty$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_n w^n &= \sum_{n=0}^{\infty} w^n \sum_{k=n}^{\infty} {}^kC_n \, a_k \alpha^{k-n} \\ &= \sum_{k=0}^{\infty} a_k \sum_{n=0}^{k} {}^kC_n \, w^n \alpha^{k-n} \\ &= \sum_{k=0}^{\infty} a_k (w + \alpha)^k\end{aligned}$$

by interchanging the order of summation. ∎

2.4 The Complex Logarithm

We would like the logarithm to be the inverse function of the exponential. The exponential function never takes the value zero. This is a consequence of

$$\exp(z)\exp(-z) = \exp(z - z) = \exp(0) = 1$$

so let $z \in \mathbb{C} \setminus \{0\}$. Let $z = r\,\mathrm{cis}(\theta)$. Then the equation $\exp(w) = z$ can be solved. If $w = u + iv$ with u and v real, we get $e^u = r$ and $\mathrm{cis}(v) = \mathrm{cis}(\theta)$. Hence $u = \ln(r)$ and $\theta = v + 2n\pi i$ where $n \in \mathbb{Z}$. So, unfortunately there are infinitely many solutions. There are three ways of proceeding. The politician's solution to this situation is to do nothing and to assert that the complex logarithm is a multivalued function. This solution is largely unworkable. The engineer's solution is based on the observation that if you decide to choose a particular solution, say the one which would have $\log(1) = 0$, then you will run into trouble because as the point z starting at 1 makes an anticlockwise tour of the origin and comes back to 1 and if the logarithm is continuous along this path, then on returning to the origin, the value taken would be $2\pi i$. The problem arises from making a circuit of the origin and the engineer sees that the solution is to prevent the making of circuits around the origin. To do this he/she makes a cut from the origin out to infinity. The position of the cut is somewhat arbitrary, but usually it is located in the most unobtrusive location, namely along the negative real axis $N =]-\infty, 0]$. Restricted to the set $\mathbb{C} \setminus N$, the angle θ may be defined in the range $-\pi < \theta < \pi$ and the complex logarithm so obtained (and called the ***principal branch of the logarithm***) is given by

$$\log(r\,\mathrm{cis}(\theta)) = \ln(r) + i\theta$$

where indeed, $-\pi < \theta < \pi$.

The final solution to this situation (the mathematician's solution) is to assert that it is necessary to define the complex logarithm on a different space (the universal covering space of $\mathbb{C} \setminus \{0\}$) which in this context can be given the structure of a Riemann surface. We do not explore this solution at the moment, but possibly will do so later if time allows.

Since we are dealing with power series, we should consider the power series

$$f(z) = \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{z^n}{n}$$

which is easily seen to have radius 1. Note that if z is real and $-1 < z < 1$ then $f(z) = \ln(1 + z)$. This series vanishes at $z = 0$ and so the Composition

Theorem for power series applies and tells us that the function $z \mapsto \exp(f(z))$ has a power series expansion with radius at least 1. If z is real and $-1 < z < 1$ then $\exp(f(z)) = 1 + z$. Since the coefficients of a real power series are uniquely determined by the function represented by the sum, we see that the relationship $\exp(f(z)) = 1 + z$ must continue to hold for all z complex with $|z| < 1$. So $f(z)$ is a logarithm of $1 + z$, but which one? Note that if $|z| < 1$, then $\Re(z) > 0$ and it follows that the argument of $1 + z$ is in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We establish that this θ is in fact the imaginary part of $f(z)$ by continuity. Explicitly, we observe that

$$h(x, y) = \Im(f(x + iy)) - \arctan\left(\frac{y}{1+x}\right)$$

takes values in $2\pi\mathbb{Z}$ and is continuous in the open unit disk $\{(x, y); x^2 + y^2 < 1\}$ in the plane. At $(x, y) = (0, 0)$ we have that $h(0, 0) = 0$ since $f(0) = 0$ and $\arctan(0) = 0$. The function h is a continuous function of a connected space (the unit disk) into a discrete space ($2\pi\mathbb{Z}$) and hence is constant. If you have difficulty with this you can also apply the Intermediate Value Theorem to obtain the same result. To do this, you take an arbitrary point (x, y) of the unit disk and define

$$H(t) = h(tx, ty)$$

and continuous function of $[0, 1]$ into $2\pi\mathbb{Z}$. If $H(0) \neq H(1)$, then the Intermediate Value Theorem gives a contradiction.

2.5 Complex Derivatives of Power Series

LEMMA 9 *We have the identity*

$$(z + h)^n - z^n - nhz^{n-1} = h^2 \sum_{\ell=0}^{n-2} (n - \ell - 1)(z + h)^\ell z^{n-\ell-2}$$

Sketch Proof. We have by the summation formula for a geometric series

$$\sum_{\ell=0}^{n-2} (z + h)^\ell (tz)^{n-\ell-1} = (tz)^{n-1} \sum_{\ell=0}^{n-2} \left(\frac{z + h}{tz}\right)^\ell = \frac{tz(z + h)^{n-1} - (tz)^n}{(1 - t)z + h}$$

Now, differentiating both sides partially with respect to t and then setting $t = 1$ we get

$$z\sum_{\ell=0}^{n-2}(n-\ell-1)(z+h)^{\ell}z^{n-\ell-2} = \frac{hz(z+h)^{n-1} - nhz^n + z^2(z+h)^{n-1} - z^{n+1}}{h^2}$$

Multiplying by h^2z^{-1} and simplifying, now gives the result. ■

THEOREM 10 *Let the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius $\rho > 0$ and define a function f in $|z| < \rho$. Then f has a complex derivative $f'(z)$ at every point of $|z| < \rho$ and $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ and the derived power series also has radius ρ.*

Proof. It is clear from the formula for the radius of convergence that the two series have the same radius of convergence ρ. We only need to establish that $f'(z) = g(z)$ in $|z| < \rho$ where

$$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

If $|z| < \rho$, then we can find r such that $|z| < r < \rho$ and we insist that $|h| < r - |z|$, so that $|z+h| \le |z| + |h| < r$ also. We get from the lemma that

$$\begin{aligned}
\left|\frac{f(z+h) - f(z) - hg(z)}{h^2}\right| &\le \sum_{n=2}^{\infty} |a_n| \sum_{\ell=0}^{n-2}(n-\ell-1)|z+h|^{\ell}|z|^{n-\ell-2} \\
&\le \sum_{n=2}^{\infty} |a_n| r^{n-2} \sum_{\ell=0}^{n-2}(n-\ell-1) \\
&= \sum_{n=2}^{\infty} \frac{1}{2} n(n-1)|a_n| r^{n-2} = C < \infty
\end{aligned}$$

Hence we find

$$\left|\frac{f(z+h) - f(z)}{h} - g(z)\right| \le C|h|$$

and the result follows. ■

COROLLARY 11 *Let the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius $\rho > 0$ and define a function f in $|z| < \rho$. Then f has a complex derivatives $f^{(k)}(z)$ of all orders $k \in \mathbb{Z}^+$ at every point of $|z| < \rho$ and $f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}$ and each of these power series also has radius ρ.*

COROLLARY 12 *Let the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius $\rho > 0$. Then the coefficients a_n are uniquely determined by f from the formula*

$$a_n = \frac{1}{n!} f^{(n)}(0) \text{ for } n \in \mathbb{Z}^+.$$

3

A Rapid Review of Multivariable Calculus

In a single variable, differential calculus is seen as the study of limits of quotients of the type
$$\frac{f(v)-f(v_0)}{v-v_0}.$$

This approach works when the domain of the function f is one-dimensional.

DEFINITION *Let $g : \,]a,b[\, \longrightarrow V$ where V is a finite-dimensional normed real vector space. Let $t \in \,]a,b[$. Then the quotient*

$$f(s) = (s-t)^{-1}(g(s)-g(t)) \in V$$

is defined for s in $]a,b[\, \setminus \{t\}$. It is not defined at $s = t$. If

$$\lim_{s\to t} f(s)$$

*exists, then we say that g is **differentiable** at t and the value of the limit is denoted $g'(t)$ and called the **derivative** of g at t. It is an element of V.*

In several variables this approach no longer works. We need to view the derivative at v_0 as a linear map df_{v_0} such that we have

$$f(v) = f(v_0) + df_{v_0}(v - v_0) + \text{ error term.}$$

Here, the quantity $f(v)$ has been written as the sum of three terms. The term $f(v_0)$ is the constant term. It does not depend on v. The second term $df_{v_0}(v-v_0)$ is a linear function df_{v_0} of $v-v_0$. Finally the third term is the error term. The linear map df_{v_0} is called the ***differential*** of f at v_0. The differential is also called the ***Fréchet derivative*** . Sometimes we collect together the first and second terms as an ***affine*** function of v. A function is affine if and only if it is a constant function plus a linear function. This then is the key idea of differential calculus. We attempt to approximate a given function f at a given point v_0 by an affine function within an admissible error. Which functions are admissible errors for this purpose? We answer this question in the next section.

There are two settings that we can use to describe the theory. We start out using abstract real normed vector spaces. However as soon as one is faced with real problems in finitely many dimensions one is going to introduce coordinates — i.e. one selects bases in the vector spaces and works with the coordinate vectors. This leads to the second concrete setting which interprets differentials by Jacobian matrices.

3.1 The Little "o" of the Norm Class

Let V and W be finite-dimensional real normed vector spaces.

DEFINITION *Let $\Omega \subseteq V$ be an open set and let $v_0 \in \Omega$. Then a function $\varphi : \Omega \longrightarrow W$ is in the class $\mathcal{E}_{\Omega,v_0}$ called* ***little "o" of the norm*** *at v_0 iff for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\|\varphi(v)\| \leq \epsilon\|v - v_0\|$$

for all $v \in \Omega$ with $\|v - v_0\| < \delta$.

It is clear from the definition that if $\varphi \in \mathcal{E}_{\Omega,v_0}$ then $\varphi(v_0) = 0$.

If we replace the norms on V and W by equivalent norms then it is clear that the class of functions $\mathcal{E}_{\Omega,v_0}$ does not change. Since all norms on a finite dimensional real vector space are equivalent, we see that the class $\mathcal{E}_{\Omega,v_0}$ is completely independent of the norms on V and W. In other words, the class $\mathcal{E}_{\Omega,v_0}$ is an invariant of the linear space structure of V and W.

The following Lemma is very important for the definition of the differential. It tells us that we can distinguish between a linear function of $v - v_0$ and an admissible error function.

LEMMA 13 *Let $\Omega \subseteq V$ be an open set and let $v_0 \in \Omega$. Let $\varphi : \Omega \longrightarrow W$ be given by*

$$\varphi(v) = \lambda(v - v_0) \qquad \forall v \in \Omega$$

where $\lambda : V \longrightarrow W$ is a linear mapping. Suppose that $\varphi \in \mathcal{E}_{\Omega,v_0}$. Then $\varphi(v) = 0$ for all $v \in \Omega$.

Proof. Let $u \in V$. Then for all $\epsilon > 0$ we have

$$\|\varphi(v_0 + tu)\| \leq \epsilon \|tu\|$$

for all values of t such that $|t|$ is small enough. Using the specific form of φ we obtain

$$\|\lambda(tu)\| \leq \epsilon \|tu\|.$$

Using the linearity and the definition of the norm, this leads to

$$|t| \|\lambda(u)\| \leq \epsilon |t| \|u\|.$$

Choosing now t small and non-zero, we find that

$$\|\lambda(u)\| \leq \epsilon \|u\|.$$

Since this is true for all $\epsilon > 0$ we have $\lambda(u) = 0$. But this holds for all $u \in V$ and the result follows. ■

The next Proposition is routine and will be used heavily in these notes.

PROPOSITION 14 *Let $\Omega \subseteq V$ be an open set and let $v_0 \in \Omega$. Then $\mathcal{E}_{\Omega,v_0}$ is a vector space under pointwise addition and scalar multiplication.*

We leave the proof to the reader.

3.2 The Differential

In this section, U, V and W are finite-dimensional real normed vector spaces.

DEFINITION *Let $\Omega \subseteq V$ be an open set and let $v_0 \in \Omega$. Then a function $f : \Omega \longrightarrow W$ is **differentiable at** v_0 with differential df_{v_0} (a linear map from V to W) iff there exists a function $\varphi : \Omega \longrightarrow W$ in the class $\mathcal{E}_{\Omega,v_0}$ such that*

$$f(v) = f(v_0) + df_{v_0}(v - v_0) + \varphi(v) \qquad \forall v \in \Omega. \tag{3.1}$$

In this situation, the quantity df_{v_0} is called the ***differential*** of f at v_0.

It is an immediate consequence of Lemma 13 that if the derivative df_{v_0} exists then it is unique.

EXAMPLE If f is a linear mapping from V to W, then it is everywhere differentiable and its derivative is given by

$$df_{v_0}(v) = f(v).$$

The error term is zero. □

EXAMPLE If α is a bilinear mapping $\alpha : \mathbb{R}^a \oplus \mathbb{R}^b \longrightarrow \mathbb{R}^k$, then we have

$$\begin{aligned}\alpha(x, y) &= \alpha(x_0 + (x - x_0), y_0 + (y - y_0)) \\ &= \alpha(x_0, y_0) + \alpha(x_0, y - y_0) + \alpha(x - x_0, y_0) + \alpha(x - x_0, y - y_0) \quad (3.2)\end{aligned}$$

The first term in (3.2) is the constant term, the second and third terms are linear. The last term is little "o" of the norm since

$$\|\alpha(x - x_0, y - y_0)\| \le \|\alpha\|_{\text{op}}\|x - x_0\|\|y - y_0\|.$$

Here $\| \ \|_{\text{op}}$ stands for the bilinear operator norm. □

We use the notation $U(v, t)$ for a $t > 0$ and v a vector in a finite-dimensional vector space V to designate the open ball centred at v of radius t. In symbols

$$U(v, t) = \{w \in V; \|w - v\| < t\}$$

PROPOSITION 15 *Let $\Omega \subseteq V$ be an open set and let $f : \Omega \longrightarrow W$ be a function differentiable at $v_0 \in \Omega$. Then f is **Lipschitz at** v_0 in the sense that there exists $\delta > 0$ and $0 < C < \infty$ such that*

$$\|f(v) - f(v_0)\| \le C\|v - v_0\|$$

whenever $v \in \Omega \cap U(v_0, \delta)$. In particular, f is continuous at v_0.

Proof. Using the notation of (3.1), we have

$$\|df_{v_0}(v - v_0)\| \le \|df_{v_0}\|_{\text{op}}\|v - v_0\| \quad (3.3)$$

and for $\epsilon = 1$, there exists $\delta > 0$ such that

$$\|\varphi(v)\| \le \|v - v_0\|. \quad (3.4)$$

for $v \in \Omega \cap U(v_0, \delta)$. Combining (3.3) and (3.4) with (3.1) we find

$$\|f(v) - f(v_0)\| \le (\|df_{v_0}\|_{\text{op}} + 1)\|v - v_0\|$$

for $v \in \Omega \cap U(v_0, \delta)$ as required. ■

Proposition 15 has a partial converse.

PROPOSITION 16 *Let $\Omega \subseteq V$ be an open set and let $f : \Omega \longrightarrow W$ be a function differentiable at $v_0 \in \Omega$. Suppose that there exists $\delta > 0$ and $0 < C < \infty$ such that*

$$\|f(v) - f(v_0)\| \le C\|v - v_0\|$$

whenever $v \in \Omega \cap U(v_0, \delta)$. Then $\|df_{v_0}\|_{\mathrm{op}} \le C$.

Proof. We write

$$f(v) = f(v_0) + df_{v_0}(v - v_0) + \varphi(v - v_0)$$

where $\varphi \in \mathcal{E}_{\Omega,v_0}$. Let $\epsilon > 0$. The, there exists δ_1 with $0 < \delta_1 < \delta$ such that

$$v \in \Omega, \|v - v_0\|_V < \delta_1 \Longrightarrow \|\varphi(v - v_0)\|_W \le \epsilon\|v - v_0\|_V$$

and consequently, for $v \in \Omega$ with $\|v - v_0\|_V < \delta_1$ we find

$$\|df_{v_0}(v - v_0)\|_W \le (C + \epsilon)\|v - v_0\|_V.$$

Since $v - v_0$ is free to roam in a ball centered at 0_V, it follows that $\|df_{v_0}\|_{\mathrm{op}} \le C + \epsilon$. Finally, since ϵ is an arbitrary positive number, we have the desired conclusion. ■

The following technical Lemma will be needed for the Chain Rule.

LEMMA 17 *Let $\Omega \subseteq V$ be an open set, Δ an open subset of W, and let $f : \Omega \longrightarrow \Delta$ be a function Lipschitz at $v_0 \in \Omega$. Let $\psi : \Delta \longrightarrow U$ be in $\mathcal{E}_{\Delta,f(v_0)}$. Then the composed function $\psi \circ f$ is in $\mathcal{E}_{\Omega,v_0}$.*

Proof. There exists $\delta_1 > 0$ and $0 < C < \infty$ such that

$$\|f(v) - f(v_0)\| \le C\|v - v_0\| \tag{3.5}$$

whenever $v \in \Omega \cap U(v_0, \delta_1)$. Let $\epsilon > 0$. Define $\epsilon_1 = C^{-1}\epsilon > 0$. Then since ψ is little "o" of the norm, there exists $\delta_2 > 0$ such that we have

$$\|\psi(w)\| \le \epsilon_1\|w - f(v_0)\|$$

provided $w \in \Delta$ and $\|w - f(v_0)\| < \delta_2$. Now define $\delta = \min(\delta_1, C^{-1}\delta_2) > 0$. Then, using (3.5), $v \in \Omega$ and $\|v - v_0\| < \delta$ together imply that $\|f(v) - f(v_0)\| < \delta_2$ and hence also

$$\|\psi(f(v))\| \le \epsilon_1\|f(v) - f(v_0)\| \le C\epsilon_1\|v - v_0\|.$$

Since $\epsilon = C\epsilon_1$, this completes the proof. ■

Theorem 18 (Chain Rule) *Let $\Omega \subseteq V$ be an open set, Δ an open subset of W, let $f : \Omega \longrightarrow \Delta$ be a function differentiable at $v_0 \in \Omega$ and let $g : \Delta \longrightarrow U$ be differentiable at $f(v_0)$. Then the composed function $g \circ f$ is differentiable at v_0 and*

$$d(g \circ f)_{v_0} = dg_{f(v_0)} \circ df_{v_0}.$$

Proof. We use the differentiability hypotheses to write

$$f(v) = f(v_0) + df_{v_0}(v - v_0) + \varphi(v) \qquad \forall v \in \Omega \tag{3.6}$$

and

$$g(w) = g(f(v_0)) + dg_{f(v_0)}(w - f(v_0)) + \psi(w) \qquad \forall w \in \Delta \tag{3.7}$$

where $\varphi : \Omega \longrightarrow W$ is in the class $\mathcal{E}_{\Omega,v_0}$ and $\psi : \Delta \longrightarrow U$ is in the class $\mathcal{E}_{\Delta,f(v_0)}$. Combining (3.6) and (3.7) yields

$$g(f(v)) = g(f(v_0)) + dg_{f(v_0)}(df_{v_0}(v - v_0) + \varphi(v)) + \psi(f(v)) \qquad \forall v \in \Omega.$$

Using the linearity of $dg_{f(v_0)}$ we can rewrite this in the form

$$g \circ f(v) = g \circ f(v_0) + (dg_{f(v_0)} \circ df_{v_0})(v - v_0) + dg_{f(v_0)}(\varphi(v)) + \psi(f(v)), \tag{3.8}$$

for all $v \in \Omega$. The first term on the right of (3.8) is constant and the second term is linear because it is the composition of two linear functions. Since $\mathcal{E}_{\Omega,v_0}$ is a vector space, it suffices to show that the third and fourth terms on the right of (3.8) are in $\mathcal{E}_{\Omega,v_0}$. For $dg_{f(v_0)}(\varphi(v))$ this is a consequence of the continuity of $dg_{f(v_0)}$, and for $\psi(f(v))$ it is a consequence of Lemma 17. ■

There is no product rule as such in the multivariable calculus, because it is not clear which product one should take.

Example For the most general case of the product rule, α is a bilinear mapping $\alpha : \mathbb{R}^a \times \mathbb{R}^b \longrightarrow \mathbb{R}^k$. Let now Ω be open in V and let $x_0 \in \Omega$. Let f and g be mappings from Ω into $\mathbb{R}^a$ and $\mathbb{R}^b$ respectively differentiable at x_0. Then let

$$h(x) = \alpha(f(x), g(x)) \qquad \forall x \in \Omega.$$

Applying the chain rule and using the derivative of α found earlier, we find that h is differentiable at x_0 and the derivative is given by

$$dh_{x_0}v = \alpha(f(x_0), dg_{x_0}v) + \alpha(df_{x_0}v, g(x_0)).$$

□

3.3 Derivatives, Differentials, Directional and Partial Derivatives

We have already seen how to define the derivative of a vector valued function on page 22. How does this definition square with the concept of differential given in the last chapter? Let V be a general normed vector space, $g :]a, b[\longrightarrow V$ and t a point of $]a, b[$. Then, it follows directly from the definitions of derivative and differential that the existence of one of $f'(t)$ and df_t implies the existence of the other, and

$$df_t(1) = f'(t).$$

This formula reconciles the fundamental difference between $f'(t)$ and df_t, namely that $f'(t)$ is a vector and df_t is a linear transformation. In effect, the existence of the limit

$$f'(t) = \lim_{s \to t}(s - t)^{-1}(f(s) - f(t))$$

as an element of V, is the same as showing that the quantity

$$f(s) - (f(t) + (s - t)f'(t))$$

is little "o" of $s - t$. Thus, $df_t(s - t) = (s - t)f'(t)$ or equivalently $df_t(1) = f'(t)$.

For a one-dimensional domain, the concepts of derivative and differential are closely related. We can attempt to understand the case in which the domain is multidimensional by restricting the function to lines. Let us suppose that Ω be an open subset of a normed vector space U and that $u_0 \in \Omega, u_1 \in U$. We can then define a function $g : \mathbb{R} \longrightarrow U$ by $g(t) = u_0 + tu_1$. The function g parametrizes a line through u_0. We think of u_1 as the ***direction vector***, but this term is a misnomer because the magnitude of u_1 will play a role. For $|t|$ small enough, $g(t) \in \Omega$. Hence, if $f : \Omega \longrightarrow V$ is a differentiable function, the composition $f \circ g$ will be differentiable in some neighbourhood of 0 and

$$(f \circ g)'(0) = d(f \circ g)_0(1) = df_{u_0}dg_0(1) = df_{u_0}g'(0) = df_{u_0}(u_1). \qquad (3.9)$$

since both g and $f \circ g$ are defined on a one-dimensional space. Equation (3.9) allows us to understand what $df_{u_0}(u_1)$ means, but unfortunately it cannot be used to define the differential.

DEFINITION *The* ***directional derivative*** $D_{u_1}f(u_0)$ *of the function* f *at the point* u_0 *in the direction* u_1 *is defined as the value of* $(f \circ g)'(0)$ *if this exists. In symbols*

$$D_{u_1}f(u_0) = \lim_{s \to 0} s^{-1}(f(u_0 + su_1) - f(u_0)). \qquad (3.10)$$

Clearly, in case f is differentiable, we can combine (3.9) and (3.10) to obtain

$$df_{u_0}(u_1) = D_{u_1} f(u_0). \tag{3.11}$$

EXAMPLE Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

It is easy to check that f is linear on every line passing through the origin $(0, 0)$. Hence the directional derivative $D_{(\xi,\eta)} f(0, 0)$ exists for every direction vector $(\xi, \eta) \in \mathbb{R}^2$. In fact, it comes as no surprise that

$$D_{(\xi,\eta)} f(0, 0) = \begin{cases} \frac{\xi^2 \eta}{\xi^2+\eta^2} & \text{if } (\xi, \eta) \neq (0, 0), \\ 0 & \text{if } (\xi, \eta) = (0, 0). \end{cases}$$

and this is *not* a linear function of (ξ, η) and therefore cannot possibly be equal to $df_{(0,0)}(\xi, \eta)$ which would necessarily have to be linear in (ξ, η). It follows from (3.9) that $df_{(0,0)}$ cannot exist. □

Let Ω be an open subset of $\mathbb{R}^m$. Faced with a mapping $f : \Omega \longrightarrow \mathbb{R}^k$, we will typically write this mapping as

$$\begin{aligned} &f(x_1, x_2, \ldots, x_m) \\ &= (f_1(x_1, x_2, \ldots, x_m), f_2(x_1, x_2, \ldots, x_m), \ldots, f_k(x_1, x_2, \ldots, x_m)) \end{aligned}$$

where f_1 through f_k denote the corresponding coordinate functions. Then, the existence of all the partials $\partial f_i / \partial x_j$ as i runs over 1 to k is equivalent to the existence of the directional derivative $D_{e_j} f$. In case that f is differentiable at x, the $k \times m$ ***Jacobian matrix***

$$\left(\frac{\partial f_i}{\partial x_j}(x) \right)_{ij}$$

is precisely the matrix representing the linear transformation df_x with respect to the usual bases in $\mathbb{R}^m$ and $\mathbb{R}^k$. Symbolically we have

$$\begin{pmatrix} f_1(x+\xi) \\ f_2(x+\xi) \\ \vdots \\ f_k(x+\xi) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_k(x) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_m}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \cdots & \frac{\partial f_2}{\partial x_m}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1}(x) & \cdots & \frac{\partial f_k}{\partial x_m}(x) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{pmatrix} + \text{ error term,}$$

where the error term is little "o" of $\{\sum_{j=1}^m \xi_j^2\}^{\frac{1}{2}}$.

LEMMA 19 *The existence and continuity of all the partials* $\dfrac{\partial f_i}{\partial x_j}$ *for* $(1 \leq i \leq k, 1 \leq j \leq m)$ *implies the existence and continuity of the differential* df_x *for all* $x \in \Omega$.

Sketch proof. The case $k = 1, m = 2$ is entirely typical and captures the idea of the proof. We have

$$\begin{aligned}
&f_1(x_1 + \xi_1, x_2 + \xi_2) - f_1(x_1, x_2) \\
&= \Big(f_1(x_1 + \xi_1, x_2 + \xi_2) - f_1(x_1 + \xi_1, x_2)\Big) + \Big(f_1(x_1 + \xi_1, x_2) - f_1(x_1, x_2)\Big) \\
&= \frac{\partial f_1}{\partial x_2}(x_1 + \xi_1, x_2 + t_2\xi_2)\xi_2 + \frac{\partial f_1}{\partial x_1}(x_1 + t_1\xi_1, x_2)\xi_1
\end{aligned}$$

by the Mean Value Theorem and where $0 \leq t_1, t_2 \leq 1$,

$$= \frac{\partial f_1}{\partial x_2}(x_1, x_2)\xi_2 + \frac{\partial f_1}{\partial x_1}(x_1, x_2)\xi_1 + o(\|(\xi_1, \xi_2)\|)$$

since for example the difference

$$\frac{\partial f_1}{\partial x_2}(x_1 + \xi_1, x_2 + t_2\xi_2)\xi_2 - \frac{\partial f_1}{\partial x_2}(x_1, x_2)\xi_2$$

is $o(|\xi_2|)$ since

$$\frac{\partial f_1}{\partial x_2}(x_1 + \xi_1, x_2 + t_2\xi_2) - \frac{\partial f_1}{\partial x_2}(x_1, x_2)$$

tends to zero as $(\xi_1, \xi_2) \longrightarrow (0, 0)$ by the continuity of $\dfrac{\partial f_1}{\partial x_2}$. ■

3.4 Complex Derivatives and the Cauchy–Riemann Equations

Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \longrightarrow \mathbb{C}$. Using the standard identification of $\mathbb{C}$ with $\mathbb{R}^2$, we can write $f = u + iv$, where u and v are real-valued functions on Ω and we write the variable z as $x + iy$ with x and y real. In this way, we may equate $f(z) = u(x, y) + iv(x, y)$. The equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.12}$$

are known as the Cauchy–Riemann equations. The Cauchy–Riemann equations are a necessary and sufficient condition that the Jacobian matrix

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

is the matrix representing a complex multiplication.

THEOREM 20

(i) *If the function f possesses a complex derivative at a point, then the Fréchet derivative exists at that point and is a complex multiplication.*

(ii) *Conversely, if f has a Fréchet derivative derivative at a point which is a complex multiplication, then f possesses a complex derivative at the point.*

(iii) *If the function f possesses a complex derivative in Ω, then the partials $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ exist in Ω and satisfy (3.12).*

(iv) *If the partials $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ exist and are continuous in Ω and satisfy (3.12), then f has a complex derivative in Ω and the derivative is continuous in Ω.*

Proof.

(i) Since f possesses a complex derivative, say at ζ, then for h complex, we have

$$\frac{f(\zeta + h) - f(\zeta)}{h} \longrightarrow f'(\zeta) \tag{3.13}$$

as $h \longrightarrow 0$. Multiplying by h we see that

$$f(\zeta + h) = f(\zeta) + f'(\zeta)h + \varphi(h) \tag{3.14}$$

where $|\varphi(h)|$ is $o(|h|)$. So the Fréchet derivative exists at ζ. Furthermore the linear mapping $h \mapsto f'(\zeta)h$ is a complex multiplication.

(ii) Conversely, if f possesses a Fréchet derivative A at ζ, then

$$f(\zeta + h) = f(\zeta) + A(h) + \varphi(h)$$

where $|\varphi(h)|$ is $o(|h|)$. But A is also a complex multiplication, then A is multiplication by a complex number which we will designate $f'(z)$. We now have (3.14) and can divide by h to obtain (3.13).

(iii) Follows immediately from (i).

(iv) Since the partials are all continuous in Ω, it follows from Lemma 19 that f has a Fréchet derivative in Ω and the derivative is continuous in Ω. But, now we apply (ii) to see that in fact f has a complex derivative at each point of Ω

■

Next, we observe that if u and v satisfy the Cauchy–Riemann equations and have more regularity, then they are harmonic. before we can establish this, we need the symmetry of the second derivative.

LEMMA 21 *If* f, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial^2 f}{\partial x \partial y}$ *and* $\dfrac{\partial^2 f}{\partial y \partial x}$ *all exist and are continuous in an open subset* Ω *of the plane, then* $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ *in* Ω.

Proof. Let h and k be non-zero real numbers. Then consider

$$\begin{aligned} f(x+k, y+h) - f(x+k, y) - f(x, y+h) + f(x, y) \\ = k\left(\frac{\partial f}{\partial x}(\xi, y+h) - \frac{\partial f}{\partial x}(\xi, y)\right) \end{aligned} \tag{3.15}$$

from applying the Mean Value Theorem to the function g given by

$$g(t) = f(t, y+h) - f(t, y).$$

The point ξ lies between x and $x+k$. Applying the Mean Value Theorem a second time gives

$$f(x+k, y+h) - f(x+k, y) - f(x, y+h) + f(x, y) = kh\left(\frac{\partial^2 f}{\partial y \partial x}(\xi, \eta)\right).$$

where η lies between y and $y+h$. An exactly similar argument yields

$$f(x+k,y+h)-f(x+k,y)-f(x,y+h)+f(x,y)=kh(\frac{\partial^2 f}{\partial x\partial y}(\xi_1,\eta_1)),$$

where ξ_1 lies between x and $x+k$ and η_1 lies between y and $y+h$. For $kh\neq 0$ we now get

$$\frac{\partial^2 f}{\partial x\partial y}(\xi_1,\eta_1)=\frac{\partial^2 f}{\partial y\partial x}(\xi,\eta).$$

Using the continuity of both second partials at (x,y), it suffices to let k and h tend to zero to conclude that

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)=\frac{\partial^2 f}{\partial y\partial x}(x,y).$$

■

LEMMA 22 *If u and v be C^2 (i.e. possess continuous partial derivatives of all homogenous orders 0, 1 and 2) and if (3.12) holds, then*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \textit{ and } \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

Proof. We have

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}&=\frac{\partial}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\frac{\partial u}{\partial y}\\&=\frac{\partial}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)\\&=0.\end{aligned}$$

A similar argument shows that $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=0.$ ■

3.5 Exact Equations and Conjugate Harmonics

In the theory of ordinary differential equations (ODE) we meet the concept of an exact equation. This is an equation of the form

$$A(x,y)dx + B(x,y)dy = 0$$

which satisfies $\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$. We wish to find a function $F(x,y)$ such that $\dfrac{\partial F}{\partial x} = A$ and $\dfrac{\partial F}{\partial y} = B$. The exactness condition $\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$ is necessary since it ensures that $\dfrac{\partial^2 F}{\partial x \partial y} = \dfrac{\partial^2 F}{\partial y \partial x}$.

PROPOSITION 23 *Let A and B be given as above and C^1 in the rectangle $x_1 < x < x_2$, $y_1 < y < y_2$. Then a suitable C^2 primitive F can be found in the same rectangle.*

Proof. Let us assume without loss of generality that the origin lies in the rectangle. Then we define

$$F(x,y) = \int_{t=0}^{x} A(t,0)dt + \int_{s=0}^{y} B(x,s)ds$$

which is effectively a line integral along the path from $(0,0)$ to $(x,0)$ along the x-axis and then from $(x,0)$ to (x,y) parallel to the y-axis. We get

$$\frac{\partial F}{\partial y} = B(x,y)$$

directly from the Fundamental Theorem of Calculus. We also get

$$\frac{\partial F}{\partial x} = A(x,0) + \int_{s=0}^{y} \frac{\partial B}{\partial x}(x,s)ds$$

from the Fundamental Theorem of Calculus and by differentiation under $\int$

$$= A(x,0) + \int_{s=0}^{y} \frac{\partial A}{\partial y}(x,s)ds$$

by the exactness condition

$$= A(x,0) + (A(x,y) - A(x,0)) = A(x,y)$$

from the Fundamental Theorem of Calculus again. Note that a little extra work shows that F is continuous and hence, since A and B are supposed C^1, it follows that F is C^2. ■

COROLLARY 24 *Let u be a C^2 harmonic function defined in an open rectangle with sides parallel to the coordinate axes. Then there is a C^2 harmonic function v defined in the same rectangle satisfying* (3.12).

Proof. We need to solve

$$\frac{\partial v}{\partial x} = A = -\frac{\partial u}{\partial y} \text{ and } \frac{\partial v}{\partial y} = B = \frac{\partial u}{\partial x}.$$

The exactness condition is just the fact that u is harmonic. By the previous result, there is a C^2 solution v. The fact that v is harmonic follows from equating the mixed partials of u. ■

DEFINITION *Let Ω be an open subset of $\mathbb{C}$. Then $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic in Ω if it has a complex derivative $f'(z)$ at every point z of Ω and the map $z \mapsto f'(z)$ is continuous on Ω.*

If u is the real part of a holomorphic function and is also C^2, then the material above shows how to construct the imaginary part v. Such a v is called the ***conjugate harmonic*** of u.

EXAMPLE Let $u = x^4 - 6x^2y^2 + y^4$, then for v we need to solve

$$\frac{\partial v}{\partial x} = 12x^2y - 4y^3 \text{ and } \frac{\partial v}{\partial y} = 4x^3 - 12xy^2$$

and we can eyeball that $v = 4x^3y - 4xy^3 + C$ where C is a constant of integration. The corresponding holomorphic function is $u + iv = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3 + C) = (x + iy)^4 + iC$. □

EXAMPLE Let $u = \frac{1}{2}\ln(x^2 + y^2)$. Then u is a nice function away from the origin and somewhat tedious calculations show that u is harmonic away from the origin. We wish to solve

$$\frac{\partial v}{\partial x} = -\frac{y}{x^2 + y^2} \text{ and } \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

and again, we can find $v = \arctan(x^{-1}y) + C$ in $x > 0$. A solution can be found locally anywhere in $\mathbb{C} \setminus \{0\}$, but not globally in $\mathbb{C} \setminus \{0\}$. We get $u + iv = \log(x + iy) + iC$. □

There is a more sophisticated way of looking at the Cauchy–Riemann equations that uses the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

There are two ways of understanding these definitions. The first is to point out that with $dz = dx + idy$ and $d\overline{z} = dx - idy$, we get

$$\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$

so that z and $\overline{z}$ mimic a coordinate system on $\mathbb{C}$.

This is not what is actually happening however. The vectors $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial y}$ evaluated at a point (x, y) of the plane form a basis of the tangent space $M_{(x,y)}$ to the plane at that point. The dual space $M'_{(x,y)}$ is called the cotangent space and the dual basis consists of dx and dy. Both of these spaces have complexifications[1]. The vectors $\dfrac{\partial}{\partial z}, \dfrac{\partial}{\partial \overline{z}}$ form a basis in the complexification of the tangent space and the vectors dz and $d\overline{z}$ form the corresponding dual basis in the complexification of the cotangent space.

The key point is that

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial(u+iv)}{\partial x} + i\frac{\partial(u+iv)}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{1}{2}i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

so that the equation $\dfrac{\partial f}{\partial \overline{z}} = 0$ is equivalent to the Cauchy–Riemann equations.

[1]For V a real vector space, we may construct its complexification $V \oplus iV$ which becomes a complex vector space when a complex scalar multiplication is defined by $(x + iy)(u \oplus iv) = (xu - yv) \oplus i(xv + yu)$. The dimension of $V \oplus iV$ as a complex vector space is the same as the dimension of V as a real vector space. Those familiar with tensor products of vector spaces can think of the complexification as $\mathbb{C} \otimes V$, the tensor product of real vector spaces, with the complex multiplication defined by extending $\lambda(z \otimes v) = (\lambda z) \otimes v$ by linearity.

EXAMPLE The function $f(z) = |z|^2$ has a complex derivative only at the origin. Clearly f is infinitely differentiable and we check from $f(z) = z\overline{z}$ that $\dfrac{\partial f}{\partial \overline{z}} = z$. Thus, f possesses a complex derivative if and only if $z = 0$. □

It is also worth pointing out that

$$\frac{\partial^2 f}{\partial z \partial \overline{z}} = \frac{\partial^2 f}{\partial \overline{z} \partial z} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = \frac{1}{4}\triangle f.$$

A C^2 holomorphic function is necessarily harmonic! Short proof:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\frac{\partial}{\partial z}\frac{\partial f}{\partial \overline{z}} = 4\frac{\partial}{\partial z}0 = 0.$$

We finish this section by showing how Cauchy's Theorem can be obtained from Green's Theorem. Green's Theorem can be stated as

THEOREM 25 (GREEN'S THEOREM) *Let Ω be a bounded connected open subset of $\mathbb{R}^2$ such that $\partial\Omega$ consists of a finite number of piecewise smooth closed curves. Let P, Q be C^1 functions defined on an open subset containing* $\mathrm{cl}(\Omega)$. *Then*

$$\int_{\partial\Omega} P dx + Q dy = \iint\limits_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy$$

Applying this yields

$$\int_{\partial\Omega} (u + iv)(dx + idy) = \iint\limits_{\Omega} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx dy + i \iint\limits_{\Omega} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx dy$$

so that if $f = u + iv$ is holomorphic in an open subset containing $\mathrm{cl}(\Omega)$, then we have

$$\int_{\partial\Omega} f dz = 0.$$

4

Complex Integration

We want to define complex integrals along curves. In this course we will work with curves that have a piecewise C^1 parametrization. A more general theory, which we do not attempt, deals with rectifiable curves. Before we can approach this subject, we need to define the standard Riemann integral of a complex-valued function.

DEFINITION *Let $f : [a, b] \longrightarrow \mathbb{C}$ be continuous. Then we define*

$$\int_a^b f(t)dt = \int_a^b \Re f(t)dt + i \int_a^b \Im f(t)dt.$$

It is easy to see that this definition is complex linear — we check in turn that it is additive, stable under real scalar multiplication and stable under scalar multiplication by i. We have

LEMMA 26

$$\left| \int_a^b f(t)dt \right| \leq \int_a^b |f(t)|dt \tag{4.1}$$

Proof. Write $\int_a^b f(t)dt = r\omega$ where $r \geq 0$ and $|\omega| = 1$. Then

$$r = \overline{\omega} \int_a^b f(t)dt = \int_a^b \overline{\omega} f(t)dt = \Re \int_a^b \overline{\omega} f(t)dt = \int_a^b \Re(\overline{\omega} f(t))dt$$

and so

$$\left|\int_a^b f(t)dt\right| = r \le \int_a^b |\Re(\overline{\omega} f(t))|dt \le \int_a^b |\overline{\omega} f(t)|dt = \int_a^b |f(t)|dt$$

using the inequality (4.1) for real-valued functions.

We will say that a function $t \mapsto z(t)$ from $[a,b]$ to $\mathbb{C}$ is a C^1 path if it is one-to-one and continuously differentiable. The underlying set Γ of the path is the direct image $z([a,b])$. If we have two distinct points z_1 and z_2 of the underlying set then they arise uniquely from t_1, t_2 such that $z_j = z(t_j)$ for $j = 1,2$ and we can determine whether $t_1 < t_2$ or $t_2 < t_1$, i.e. we have a concept of order along the path.

Let $f : \Gamma \longrightarrow \mathbb{C}$ be continuous. Then we define the integral along the parametrized path by

$$\int_\Gamma f(z)dz = \int_a^b f(z(t))\frac{dz}{dt}(t)dt \tag{4.2}$$

and initially the integral appears to depend upon the parametrization of the path. The integral on the right is defined because the integrand is continuous (compositions and product of continuous functions are continuous). A Riemann sum for the integral on the right of (4.2) is

$$S = \sum_{k=1}^{n} f(z(\tau_k))\frac{dz}{dt}(\tau_k)\,(t_k - t_{k-1})$$

where $a = t_0 < t_1 < \cdots < t_n = b$ and $\tau_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$. There is a related Riemann sum, namely

$$S_\Gamma = \sum_{k=1}^{n} f(z(\tau_k))(z(t_k) - z(t_{k-1})).$$

By the Fundamental Theorem of Calculus (take real and imaginary parts)

$$z(t_k) - z(t_{k-1}) = \int_{t_{k-1}}^{t_k} \frac{dz}{dt}(t)dt$$

and so

$$z(t_k) - z(t_{k-1}) - \frac{dz}{dt}(\tau_k)\,(t_k - t_{k-1}) = \int_{t_{k-1}}^{t_k} \left(\frac{dz}{dt}(t) - \frac{dz}{dt}(\tau_k)\right) dt$$

Leading to

$$\left| z(t_k) - z(t_{k-1}) - \frac{dz}{dt}(\tau_k)\,(t_k - t_{k-1}) \right| \le |t_k - t_{k-1}|\omega_{\frac{dz}{dt}}(|t_k - t_{k-1}|) \quad (4.3)$$

if the partition has step less than $\delta > 0$, then this leads to

$$|S - S_\Gamma| \le \sup_{z\in\Gamma} |f(z)||b-a|\omega_{\frac{dz}{dt}}(\delta)$$

Letting $\delta \longrightarrow 0$, we have that the Riemann sum $S \longrightarrow \int_a^b f(z(t))\frac{dz}{dt}(t)dt$ and hence also $S_\Gamma \longrightarrow \int_a^b f(z(t))\frac{dz}{dt}(t)dt$.

Since the Riemann sum S_Γ can be written in the form

$$S_\Gamma = \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})$$

for node points z_k and tag points ζ_k related appropriately to the order on the curve, we see that the path integral could have been defined in terms of such Riemann sums and that our definition is independent of the C^1 parametrization used.

The definition is easily extended to curves which are obtained by joining a finite number of segments each of which has a C^1 parametrization end to end. As we have defined path integrals (with a one-to-one parametrization), the use of piecewise parametrizations is strictly necessary for path integrals around loops.

We also note that the definition (4.2) of the integral along a parametrized path is also perfectly valid for paths which are not one-to-one and indeed it is easy to see from the real change of variables formula that if $s \mapsto t(s)$ is a C^1 map of $[c, d]$ onto $[a, b]$ with $t(c) = a$ and $t(d) = b$, then

$$\int_c^d f(z(t(s)))\frac{dz(t(s))}{ds}ds = \int_a^b f(z(t))\frac{dz}{dt}(t)dt.$$

In some instances (such as in the construction of homotopic parametrized curves), the parametrization may not be one-to-one, but this is of no consequence. The purpose of insisting that parametrization be one-to-one above is merely for the purpose of establishing an intrinsic definition for an ordered curve.

EXAMPLE Let $m \in \mathbb{Z}$. We have

$$\int_\Gamma (z-\zeta)^m dz = \begin{cases} 0 & \text{if } m \neq -1 \\ 2\pi i & \text{if } m = -1 \end{cases}$$

where Γ is an anticlockwise circle centred at ζ. To see this we set $z = \zeta + re^{i\theta}$ where the constant r is the radius of the circle and θ is the parameter for the curve. Technically, since this is a closed curve, we should integrate over two pieces $[0, \pi]$ and $[\pi, 2\pi]$, but in practice integrating over $[0, 2\pi]$ will also yield the correct result. Since $dz = ire^{i\theta}d\theta$, we get

$$\begin{aligned} \int_\Gamma (z-\zeta)^m dz &= \int_0^{2\pi} r^m e^{im\theta} ire^{i\theta} d\theta \\ &= ir^{m+1} \int_0^{2\pi} \Big(\cos((m+1)\theta) + i \sin((m+1)\theta) \Big) d\theta \end{aligned}$$

which yields the stated result. □

LEMMA 27 *Let $f : \Gamma \longrightarrow \mathbb{C}$ be continuous. Then we have the estimate*

$$\left| \int_\Gamma f(z)dz \right| \leq \sup_{z \in \Gamma} |f(z)| \int_a^b \left| \frac{dz}{dt}(t) \right| dt \tag{4.4}$$

and the interpretation of $\displaystyle\int_a^b \left| \frac{dz}{dt}(t) \right| dt$ *is the pathlength of* Γ *and will be denoted* $\text{length}(\Gamma)$.

Proof.

Starting with

$$\int_\Gamma f(z)dz = \int_a^b f(z(t)) \frac{dz}{dt}(t)dt$$

we clearly get

$$\left| \int_\Gamma f(z)dz \right| \leq \int_a^b |f(z(t))| \left| \frac{dz}{dt}(t) \right| dt$$

from (4.1). The inequality (4.4) follows. ■

A Riemann sum for $\displaystyle\int_a^b \left|\frac{dz}{dt}(t)\right| dt$ would be $\displaystyle\sum_{k=1}^n \left|\frac{dz}{dt}\right|(\tau_k)\,(t_k - t_{k-1})$. By (4.3), this quantity differs from the Riemann sum $\sum_{k=1}^n |z(t_k) - z(t_{k-1})|$ by at most $|b-a|\omega_{\frac{dz}{dt}}(\delta)$, where δ is the step of the partition. We can define the pathlength of Γ as $\sup \sum_{k=1}^n |z_k - z_{k-1}|$ where z_k are points on Γ written in order along the curve, the sup being taken over all such finite sets of points. Since, from the extended triangle inequality, the quantity under the sup increases as the partition gets finer and $|b-a|\omega_{\frac{dz}{dt}}(\delta) \longrightarrow 0$, the quantity $\displaystyle\int_a^b \left|\frac{dz}{dt}(t)\right| dt$ may be interpreted as the pathlength of Γ.

4.1 Fundamental Theorem of Calculus for Holomorphic Functions

THEOREM 28 *Let Ω be an open subset of $\mathbb{C}$, let F be holomorphic in Ω. Then*

$$\int_\Gamma F'(z)dz = F(\beta) - F(\alpha)$$

where Γ is any C^1 path contained in Ω starting at α and ending at β.

Proof. Choosing a parametrization of the path Γ, we see that we must show

$$\int_a^b F'(z(t))\frac{dz}{dt}(t)\,dt = F(\beta) - F(\alpha).$$

But the chain rule gives $\dfrac{d}{dt}F(z(t)) = F'(z(t))\dfrac{dz}{dt}(t)$. The interpretation of this in terms of differentials is that $d_t(F \circ z)$ is $d_{z(t)}F \circ d_t z$, $d_t z$ is the multiplication operator from $\mathbb{R}$ to $\mathbb{C}$ defined by $\dfrac{dz}{dt}(t)$ and $d_{z(t)}F$ is the multiplication operator from $\mathbb{C}$ to $\mathbb{C}$ defined by $F'(z(t))$. Thus, it remains to show that

$$\int_a^b \frac{d}{dt}F(z(t))\,dt = F(\beta) - F(\alpha).$$

and now it suffices to take real and imaginary parts and apply the real variable Fundamental Theorem of Calculus. ■

As in the real variable situation, there is a second version of the Fundamental Theorem. This is more subtle however.

DEFINITION *An open subset Ω of $\mathbb{C}$ is* ***star shaped*** *at ζ if and only if for every point z of Ω, then line segment joining ζ to z lies entirely in Ω.*

THEOREM 29 *Let Ω be an open subset of $\mathbb{C}$ which is star shaped at ζ. Let f be holomorphic in Ω with complex derivative f'. Let*

$$F(z) = \int_\zeta^z f(w)dw$$

the integral being taken along the line segment from ζ to z. Then F is holomorphic in Ω and $F'(z) = f(z)$ for all $z \in \Omega$.

Proof. We may parametrize the line segment by $t \mapsto \zeta + t(z-\zeta)$ as t runs from 0 to 1. So

$$F(z) = \int_0^1 f(\zeta + t(z-\zeta))(z-\zeta)dt$$

We differentiate under the integral sign (possible since the integrand is a C^1 function of z) to get

$$\frac{\partial F}{\partial \overline{z}} = \int_0^1 \frac{\partial}{\partial \overline{z}}\Big(f(\zeta + t(z-\zeta))(z-\zeta)\Big)dt = \int_0^1 0\,dt = 0$$

and

$$\begin{aligned}\frac{\partial F}{\partial z} &= \int_0^1 \frac{\partial}{\partial z}\Big(f(\zeta + t(z-\zeta))(z-\zeta)\Big)dt \\ &= \int_0^1 f(\zeta + t(z-\zeta)) + f'(\zeta + t(z-\zeta))t(z-\zeta)dt \\ &= \int_0^1 \frac{\partial}{\partial t}\Big(tf(\zeta + t(z-\zeta))\Big)dt \\ &= \Big[tf(\zeta + t(z-\zeta))\Big]_0^1 = f(z)\end{aligned}$$

It will follow from these identities that $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial F}{\partial y}$ are both continuous in Ω. So the Fréchet derivative of F exists in Ω. Thus F is holomorphic and $F'(z) = f(z)$ in Ω. ∎

COROLLARY 30 *Let Ω be an open subset of $\mathbb{C}$ which is star shaped at ζ. Let f be holomorphic in Ω with complex derivative f'. Let*

$$F(z) = \int_\zeta^z f(w)dw$$

the integral being taken along any C^1 path from ζ to z lying in Ω. Then F is holomorphic in Ω and $F'(z) = f(z)$ for all $z \in \Omega$.

Proof. First define $F(z)$ by the integral along the line segment, as in Theorem 29. Then since F is holomorphic, $F'(z) = f(z)$, we can apply Theorem 28. Taking account of $F(\zeta) = 0$, we get the desired conclusion. ■

This now gives us a version of Cauchy's Theorem

COROLLARY 31 *Let Ω be an open subset of $\mathbb{C}$ which is star shaped at ζ. Let f be holomorphic in Ω. Let Γ be a piecewise C^1 loop passing thru ζ. Then*

$$\int_\Gamma f(z)dz = 0.$$

Proof. Choose a point ζ_1 on the loop different from ζ. Let Γ_1 be the path from ζ to ζ_1 going forward along the loop and Γ_2 be the path from ζ to ζ_1 going backwards along the loop. According to the previous corollary

$$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$$

and so

$$\int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz - \int_{\Gamma_2} f(z)dz = 0.$$

■

We can generalize this result to loops that are nicely homotopic.

THEOREM 32 *Let Ω be an open subset of $\mathbb{C}$. Let $(s,t) \mapsto z(s,t)$ be a mapping of the rectangle $[0,1] \times [a,b]$ into Ω continuous on the given rectangle. Let z_k for $k = 1, \ldots, n$ be the restriction of z to strips $[0,1] \times [t_{k-1}, t_k]$. Suppose that z_k, $\dfrac{\partial z_k}{\partial t}$, $\dfrac{\partial z_k}{\partial s}$, $\dfrac{\partial^2 z_k}{\partial s \partial t}$, $\dfrac{\partial^2 z_k}{\partial t \partial s}$ are all continuous on the strip $[0,1] \times [t_{k-1}, t_k]$ on* *which it is defined. Suppose that $z(s,a) = z(s,b)$ for all $s \in [0,1]$. Let $\Gamma(s)$ be the piecewise continuous loop defined by $t \mapsto z(s,t)$. Then $\displaystyle\int_{\Gamma(s)} f(z)dz$ is independent of s and consequently*

$$\int_{\Gamma(1)} f(z)dz = \int_{\Gamma(0)} f(z)dz.$$

Proof. We consider

$$\begin{aligned}
\frac{\partial}{\partial s}\int_{\Gamma(s)} f(z)dz &= \frac{\partial}{\partial s}\int_a^b f(z(s,t))\frac{\partial z}{\partial t}(s,t)dt \\
&= \int_a^b \left(f'(z(s,t))\frac{\partial z}{\partial s}(s,t)\frac{\partial z}{\partial t}(s,t) + f(z(s,t))\frac{\partial^2 z}{\partial s \partial t}(s,t)\right) dt \\
&= \int_a^b \frac{\partial}{\partial t}\left(f(z(s,t))\frac{\partial z}{\partial s}(s,t)\right) dt \\
&= f(z(s,b))\frac{\partial z}{\partial s}(s,b) - f(z(s,a))\frac{\partial z}{\partial s}(s,a) = 0
\end{aligned}$$

since $z(s,b) = z(s,a)$ and $\dfrac{\partial z}{\partial s}(s,b) = \dfrac{\partial z}{\partial s}(s,a)$. A detailled proof requires splitting up the range of integration $[a,b]$ into its constituent intervals $[t_{k-1}, t_k]$ and applications of differentiation under $\int$ and the Fundamental Theorem of Calculus on each constituent interval. ■

In case that $t \mapsto z(0,t)$ is constant, we get

$$\int_{\Gamma(1)} f(z)dz = 0.$$

Why is this an extension of Cauchy's Theorem for star shaped regions? Well, given a region Ω star shaped at ζ and a piecewise C^1 loop $t \mapsto z(t)$ in Ω, it suffices to consider $(s,t) \mapsto (1-s)\zeta + sz(t)$. The star shaped condition implies that $(1-s)\zeta + sz(t) \in \Omega$ for all s with $0 \le s \le 1$ and all t and we observe that for $s = 0$, $(1-s)\zeta + sz(t) = \zeta$ and for $s = 1$, $(1-s)\zeta + sz(t) = z(t)$.

It is awkward to deal with C^1 homotopies. Usually homotopies are required only to be continuous. We settle this with the following technical theorem.

THEOREM 33 *Let $(s,t) \mapsto z(s,t)$ be a continuous map from $[0,1] \times [a,b]$ to the open subset Ω of $\mathbb{C}$ such that $z(s,a) = z(s,b)$ for all $s \in [0,1]$. Suppose that $t \mapsto z(0,t)$ and $t \mapsto z(1,t)$ are piecewise C^1 loops Γ_0 and Γ_1. Let f be holomorphic in Ω. Then*

$$\int_{\Gamma(1)} f(z)dz = \int_{\Gamma(0)} f(z)dz.$$

Proof. Without loss of generality, we can always assume that $a = 0$ and $b = 1$. The ambient space of the t variable is then the interval $[0,1]$ with the points 0 and 1 identified, in other words, the circle. We realize the circle in the form $\mathbb{R}/\mathbb{Z}$ and treat it as a quotient group. The group operations $+$ and $-$ that appear in the integration formulæ below are to be taken in this group, i.e. modulo 1. Since the image of z is compact and $\mathbb{C} \setminus \Omega$ is closed it follows that there is some wiggle room say $r > 0$ between the two sets. Formally

$$r = \inf_{s,t} \operatorname{dist}_{\mathbb{C}\setminus\Omega}(z(s,t)) > 0$$

because the minimum is attained. Since z is uniformly continuous, there exists $1 > \delta > 0$ such that

$$|s_1 - s_2| < \delta, |t_1 - t_2| < \delta \Longrightarrow |z(s_1,t_1) - z((s_2,t_2)| < r$$

Next, for $0 < u < \frac{1}{2}$, let φ_u be a nonnegative function on $\mathbb{R}/\mathbb{Z}$ zero outside a interval of halflength u about 0, with integral 1 and continuously differentiable[1]. We can easily build φ_u from parabolic segments as follows

$$\varphi_u(x) = \begin{cases} 2u^{-3}(x+u)^2 & \text{for } -u \le x \le -\frac{1}{2}u, \\ u^{-1} - 2u^{-3}x^2 & \text{for } -\frac{1}{2}u \le x \le \frac{1}{2}u, \\ 2u^{-3}(x-u)^2 & \text{for } \frac{1}{2}u \le x \le u, \\ 0 & \text{otherwise} \end{cases}$$

Now construct a new homotopy by[2]

$$w(s,t) = \int_0^1 \varphi_{s(1-s)\delta}(x) z(s,t-x)dx = \int_0^1 \varphi_{s(1-s)\delta}(t-x) z(s,x)dx$$

[1]The family $(\varphi_u)_{u>0}$ is an example of a summability kernel on the circle group as $u \longrightarrow 0+$. My MATH 355 notes give more detail on this topic.

[2]This is an example of a convolution integral. We can write the formula as $w_s = \varphi_{s(1-s)\delta} \star z_s$ with $*$ denoting the convolution product. My MATH 355 notes give more detail on this topic.

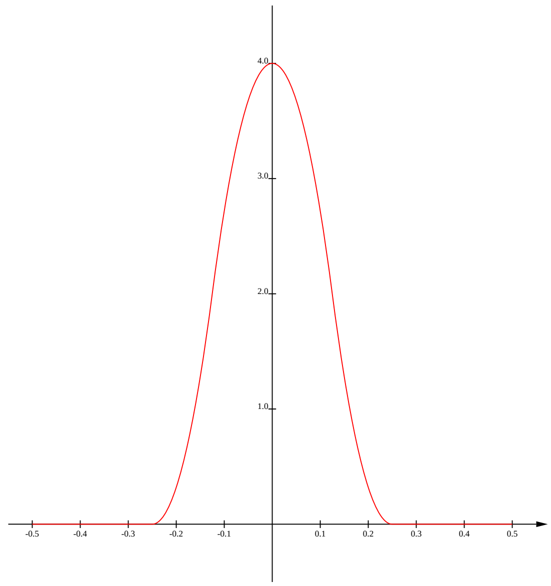

Figure 4.1: The function φ_u for $u = 0.25$.

for $0 < s < 1$, a change of variables showing that the two expressions are equal and by $w(0,t) = z(0,t)$ and $w(1,t) = z(1,t)$ for the cases $s = 0, 1$. Differentiating under the integral sign (in t) in the second expression, we see that $t \mapsto w(t,s)$ is C^1 for each $s \in [0,1]$. We leave it to the reader to check that $(s,t) \mapsto w(s,t)$ is continuous on $]0,1[\times[0,1]$. A reader familiar with more advanced material would express this by pointing out that $s \mapsto z_s$ is continuous from $]0,1[$ to $C(\mathbb{R}/\mathbb{Z})$, that $s \mapsto \varphi_{s(1-s)\delta}$ is continuous from $]0,1[$ to $L^1(\mathbb{R}/\mathbb{Z})$ and that convolution is continuous as a map $L^1(\mathbb{R}/\mathbb{Z}) \times C(\mathbb{R}/\mathbb{Z}) \longrightarrow C(\mathbb{R}/\mathbb{Z})$. A key point is that

$$w(s,t) - z(s,t) = \int_0^1 \varphi_{s(1-s)\delta}(x)(z(s,t-x) - z(s,t))dx$$

so that

$$\begin{aligned}|w(s,t) - z(s,t)| &\leq \int_0^1 \varphi_{s(1-s)\delta}(x)|z(s,t-x) - z(s,t)|dx \\ &\leq \sup_{|x|<s(1-s)\delta} |z(s,t-x) - z(s,t)| < r\end{aligned}$$

showing that $w(s,t) \in \Omega$. It remains to check that $(s,t) \mapsto w(s,t)$ is continuous on $s = 0$ and $s = 1$. Since the two cases are similar, we restrict attention to

$s = 0$. The continuity is equivalent to showing that w_s converges to w_0 uniformly on $[0, 1]$ as $s \longrightarrow 0+$. This follows since

$$w_s - w_0 = \varphi_{s(1-s)\delta} \star z_s - z_0 = \varphi_{s(1-s)\delta} \star (z_s - z_0) + (\varphi_{s(1-s)\delta} \star z_0 - z_0)$$

leading to

$$\begin{aligned} \|w_s - w_0\|_\infty &\le \|\varphi_{s(1-s)\delta}\|_1 \|z_s - z_0\|_\infty + \|\varphi_{s(1-s)\delta} \star z_0 - z_0\|_\infty \\ &\le \|z_s - z_0\|_\infty + \omega_{z_0}(s(1-s)\delta) \underset{s\to 0}{\longrightarrow} 0 \end{aligned}$$

since z_0 is uniformly continuous and uniform continuity of z implies that $z_s \longrightarrow z_0$ as $s \longrightarrow 0+$ uniformly. The new homotopy w has now been shown to satisfy all the conditions of the original homotopy z and the map $t \mapsto w(s, t)$ is C^1 for all s with $0 < s < 1$.

We will now assume that the original homotopy z has this additional property and maintain the definitions of r and δ for this new z. Then we can finish the proof quickly by showing that two C^1 loops which are sufficiently close are homotopic by a highly regular homotopy. We choose a finite sequence $0 = s_0 < s_1 < \cdots < s_n = 1$ with $s_k - s_{k-1} < \delta$ for $k = 1, 2, \ldots, n$. It is enough to show that

$$\int_{\Gamma(s_k)} f(z)dz = \int_{\Gamma(s_{k-1})} f(z)dz.$$

for $k = 1, 2, \ldots, n$. Let us fix k. Note that $|z(s_k, t) - z(s_{k-1}, t)| < r$ and it follows that the piecewise linear homotopy $w(s, t) = sz(s_k, t) + (1-s)z(s_{k-1}, t)$ (defined for $0 \le s \le 1$ and $0 \le t \le 1$) remains in Ω. But this homotopy satisfies all the regularity conditions of Theorem 32 and the result follows. ■

We are now almost ready to establish another version of Cauchy's Theorem.

DEFINITION *An open subset Ω of $\mathbb{C}$ is contractible if there is a point ζ in Ω and a continuous map $\varphi : [0, 1] \times \Omega \longrightarrow \Omega$ such that $\varphi(1, z) = z$ for all $z \in \Omega$ and$\varphi(0, z) = \zeta$ for all $z \in \Omega$.*

THEOREM 34 (CAUCHY'S THEOREM FOR CONTRACTIBLE OPEN SETS) *Let Ω be a contractible open subset of $\mathbb{C}$. Let $F : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Let Γ be a C^1 loop in Ω. Then $\displaystyle\int_\Gamma F(z)dz = 0$.*

4.2 The Winding Number

The following concept is of great importance in the sequel.

DEFINITION *Let Γ be a piecewise C^1 loop in $\mathbb{C}$. Let $\zeta \in \mathbb{C}\setminus\Gamma$, then the winding number* $\operatorname{wind}_\Gamma(\zeta)$ *is defined by*

$$\operatorname{wind}_\Gamma(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-\zeta}.$$

PROPOSITION 35 *In the above situation* $\operatorname{wind}_\Gamma(\zeta)$ *is always an integer.*

Proof. Suppose Γ is C^1 and fix a C^1 parametrization $t \mapsto z(t)$ of Γ say over $a \le t \le b$ and define

$$g(t) = \int_{s=a}^{t} \Big(z(s)-\zeta\Big)^{-1} z'(s)ds.$$

We get by applying the fundamental theorem of calculus to calculate $g'(t)$

$$\begin{aligned}\frac{d}{dt}\left\{\Big(z(t)-\zeta\Big)\exp(-g(t))\right\} &= z'(t)\exp(-g(t)) - \Big(z(t)-\zeta\Big)g'(t)\exp(-g(t))\\ &= 0,\end{aligned}$$

So, $\Big(z(t)-\zeta\Big)\exp(-g(t))$ is independent of t. Hence

$$\Big(z(a)-\zeta\Big)\exp(-g(a)) = \Big(z(b)-\zeta\Big)\exp(-g(b))$$

and since $z(a)-\zeta = z(b)-\zeta \neq 0$, we find $\exp(g(b)-g(a)) = 1$ It follows that $g(b)-g(a) \in 2\pi i\mathbb{Z}$ and the result follows. The case of a piecewise C^1 loop is only slightly more complicated. ∎

It's also necessary to understand the winding number from an intuitive point of view. Putting $z(t)-\zeta = r(t)e^{i\theta(t)}$, we get $\dfrac{dz}{dt} = \dfrac{dr}{dt}e^{i\theta} + ire^{i\theta}\dfrac{d\theta}{dt}$ and

$$\Im\Big((z(t)-\zeta)^{-1}\frac{dz}{dt}\Big) = \frac{d\theta}{dt}$$

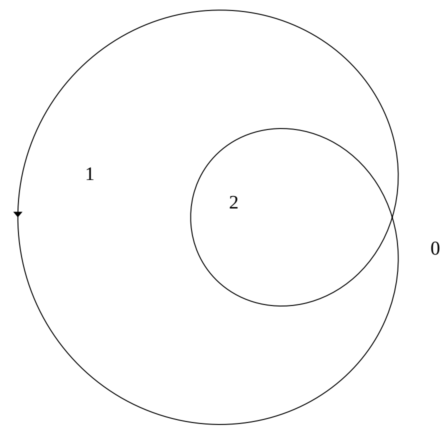

Figure 4.2: A loop and the winding number for points not on the loop.

so that

$$\operatorname{wind}_\Gamma(\zeta) = \frac{1}{2\pi}\int_a^b \frac{d\theta}{dt}dt$$

viewing θ as a multivalued function. The interpretation is that $2\pi \operatorname{wind}_\Gamma(\zeta)$ is the angle that $z(t)$ winds about ζ (in the anticlockwise sense) as t runs from a to b.

We clearly have

$$\begin{aligned}
\operatorname{wind}_\Gamma(\zeta_1) - \operatorname{wind}_\Gamma(\zeta_2) &= \frac{1}{2\pi i}\int_\Gamma \left(\frac{1}{z-\zeta_1} - \frac{1}{z-\zeta_2}\right) dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{\zeta_1 - \zeta_2}{(z-\zeta_1)(z-\zeta_2)}dz \\
|\operatorname{wind}_\Gamma(\zeta_1) - \operatorname{wind}_\Gamma(\zeta_2)| &\le \frac{|\zeta_1-\zeta_2| \operatorname{length}(\Gamma)}{2\pi \operatorname{dist}_\Gamma(\zeta_1)\operatorname{dist}_\Gamma(\zeta_2)}
\end{aligned}$$

so that $\zeta \mapsto \operatorname{wind}_\Gamma(\zeta)$ is continuous on $\mathbb{C}\setminus\Gamma$. A continuous function that takes only integer values is locally constant. For those who understand the concept of connected component, this can be expressed by saying that $\zeta \mapsto \operatorname{wind}_\Gamma(\zeta)$ is constant on each connected component of $\mathbb{C}\setminus\Gamma$.

4.3 Cauchy Integral Formula

Let Ω be open in $\mathbb{C}$. Let $\zeta_0 \in \Omega$ and let $\rho = \operatorname{dist}(\zeta_0, \mathbb{C} \setminus \Omega) > 0$. Let $0 < r < \rho$. Then the closed disk $D = \{z; |z - \zeta_0| \le r\}$ is contained in Ω. Let Γ be the circle $|z - \zeta_0| = r$ traversed anticlockwise.

THEOREM 36 *Let f be holomorphic in Ω. Then $f(\zeta) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f(z)}{z-\zeta}dz$ for $|\zeta - \zeta_0| < r$.*

Proof. The function $g(z) = \dfrac{f(z) - f(\zeta)}{z - \zeta}$ is holomorphic in $\Omega \setminus \{\zeta\}$. Let $0 < s < r - |z|$ and we imagine that s is very small. It is easy to see that Γ is homotopic to the circular loop of radius s about ζ in the region $\Omega \setminus \{\zeta\}$. Therefore

$$\int_\Gamma g(z)dz = \int_{\theta=0}^{2\pi} \frac{f(\zeta + se^{i\theta}) - f(\zeta)}{se^{i\theta}} ise^{i\theta} d\theta$$

and we get

$$\begin{aligned}\left|\int_\Gamma g(z)dz\right| &\le \int_{\theta=0}^{2\pi} \left|f(\zeta + se^{i\theta}) - f(\zeta)\right| d\theta \\ &\le 2\pi\omega_{f|_D}(s)\end{aligned}$$

But f is uniformly continuous on D, so letting $s \to 0$ gives $\displaystyle\int_\Gamma g(z)dz = 0$. We now get

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-\zeta}dz = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{z-\zeta}dz = f(\zeta)$$

as required. ∎

This result has far reaching consequences for

$$\frac{1}{z-\zeta} = \frac{1}{(z-\zeta_0) - (\zeta - \zeta_0)} = \sum_{n=0}^{\infty} (z-\zeta_0)^{-n-1}(\zeta-\zeta_0)^n$$

We note that if $|z - \zeta_0| = r$, $|\zeta - \zeta_0| \le u < r$, then

$$\left|f(z)(z-\zeta_0)^{-n-1}(\zeta-\zeta_0)^n\right| \le \sup_{z\in D}|f(z)|r^{-1}\left(\frac{u}{r}\right)^n$$

It follows that

$$f(\zeta)=\sum_{n=0}^{\infty}a_n(\zeta-\zeta_0)^n \tag{4.5}$$

with uniform and absolute convergence on $|\zeta-\zeta_0|\le u$ where

$$a_n=\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\zeta_0)^{n+1}}dz$$

and $|a_n|\le \sup_{z\in D}|f(z)|r^{-n}$. Given any fixed u with $|u|<\rho$, we may always choose r with $u<r<\rho$ and hence (4.5) must have radius at least ρ. Uniqueness considerations for the coefficients of power series guarantee that the a_n derived from two different values of r would have to be the same.

Effectively then, Theorem 36 has the following corollaries.

COROLLARY 37 *Let Ω be open in $\mathbb{C}$ and let f be holomorphic in Ω. Let $\zeta_0\in\Omega$ and let $\rho=\operatorname{dist}(\zeta_0,\mathbb{C}\setminus\Omega)>0$. Then there exist complex numbers $(a_n)_{n=0}^{\infty}$ such that the power series*

$$\sum_{n=0}^{\infty}a_n(\zeta-\zeta_0)^n$$

has radius at least ρ and converges to $f(\zeta)$ in $|\zeta-\zeta_0|<\rho$.

In particular, the concepts analytic and holomorphic are equivalent.

COROLLARY 38 *Let Ω be open in $\mathbb{C}$. Let $\zeta_0\in\Omega$ and let $\rho=\operatorname{dist}(\zeta_0,\mathbb{C}\setminus\Omega)>0$. Let $0<r<\rho$. Let Γ be the circle $|z-\zeta_0|=r$ traversed anticlockwise and $n\in\mathbb{Z}^+$. Then*

$$f^{(n)}(\zeta)=\frac{n!}{2\pi i}\int_\Gamma\frac{f(z)}{(z-\zeta)^{n+1}}dz.$$

Proof. It suffices to differentiate n times with respect to ζ the result of Theorem 36 under the integral sign. In the special case $\zeta=\zeta_0$, it can also be deduced from the formula for the coefficients of a power series in terms of the derivatives of the function it represents at the central point. ■

COROLLARY 39 (CAUCHY'S ESTIMATE) *Let Ω be open in $\mathbb{C}$. Let $\zeta_0\in\Omega$ and let $\rho=\operatorname{dist}_{\mathbb{C}\setminus\Omega}(\zeta_0)>0$. Let $0<r<\rho$. Let $n\in\mathbb{Z}^+$. Then*

$$\left|f^{(n)}(\zeta_0)\right|\le\frac{n!}{r^n}\sup_{|z-\zeta_0|=r}|f(z)|.$$

5

Holomorphic Functions – Beyond Cauchy's Theorem

In the last chapter, we saw that as soon as a function has a continuous complex derivative in an open set, then locally about every point in the open set, it has a power series expansion about that point with strictly positive radius of convergence. The corresponding result in the real setting is far from being true. In fact there are infinitely differentiable functions that do not have a power series expansion at a point. The most well known example is

$$\varphi(x) = \begin{cases} \exp(x^{-2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is infinitely differentiable on the whole of $\mathbb{R}$ and has $\varphi^{(n)}(0) = 0$ for all $n \in \mathbb{Z}^{+}$[1]. If φ had a power series expansion about 0 it would have to be

$$0 + 0x + 0x^2 + \cdots$$

and it is clear that φ is not identically zero in any neighbourhood of 0. This example does not work in $\mathbb{C}$ since $\exp(z^{-2})$ tends properly to infinity as z tends to zero along the imaginary axis. The corresponding function is not even continuous at 0.

So, holomorphic functions are very special and have many unexpected properties that we try to investigate in this chapter.

[1] To prove this show by induction that $\varphi^{(n)}(x) = \begin{cases} p_n(x^{-1}) \exp(x^{-2}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ for all $n \in \mathbb{Z}^+$ where p_n is a polynomial.

5.1 Zeros of Holomorphic Functions

The situation with regard to zeros is summarized by the following theorem.

THEOREM 40 *Let f be holomorphic in an open subset Ω of $\mathbb{C}$. Let $\zeta \in \Omega$. Then there is a "number" $m \in \mathbb{Z}^+ \cup \{\infty\}$ called the* ***order of*** ζ ***as a zero of*** f *such that*

- $m = 0$ *if f does not have a zero at ζ, i.e. $f(\zeta) \neq 0$.*
- $m = \infty$ *if f vanishes identically in a neighbourhood of ζ.*
- $m \in \mathbb{N}$ *if* $g(z) = \begin{cases} (z-\zeta)^{-m} f(z) & \text{if } z \neq \zeta, \\ \dfrac{1}{m!} f^{(m)}(\zeta) & \text{if } z = \zeta. \end{cases}$ *defines a holomorphic function in Ω and $g(\zeta) \neq 0$.*

Proof. The assertion of the Theorem is that one of the cases listed must necessarily hold. By Corollary 37 we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\zeta)^n$$

valid in $|z-\zeta| < r$ for some $r > 0$. If $a_n = 0$ for all $n \in \mathbb{Z}^+$ then we are clearly in case $m = \infty$. So, if we are not in this case, there exists n such that $a_n \neq 0$. Now the nonempty subset $\{n; a_n \neq 0\}$ of $\mathbb{Z}^+$ has a least element by the Well Ordering Principle. Let this least element be m. If $m = 0$, then $f(\zeta) = a_0 \neq 0$ and f does not have a zero at ζ. So we are left with the case $m \in \mathbb{N}$ and then $f(\zeta) = a_0 = 0$ so that f does have a zero at ζ. We consider the function defined by

$$g(z) = \begin{cases} \displaystyle\sum_{n=m}^{\infty} a_n (z-\zeta)^{n-m} & \text{if } |z-\zeta| < r, \\ (z-\zeta)^{-m} f(z) & \text{if } z \in \Omega \setminus \{\zeta\}. \end{cases}$$

We note that $\displaystyle\sum_{n=m}^{\infty} a_n (z-\zeta)^{n-m}$ agrees with $(z-\zeta)^{-m} f(z)$ in $0 < |z-\zeta| < r$ since $a_0, \ldots, a_{m-1}$ all vanish. The function $z \mapsto \displaystyle\sum_{n=m}^{\infty} a_n (z-\zeta)^{n-m}$ is holomorphic in $|z-\zeta| < r$ because it is given by a convergent power series expansion.

The function $z \mapsto (z-\zeta)^{-m} f(z)$ is holomorphic in $\Omega \setminus \{\zeta\}$ since it is the product of two holomorphic functions in that open set (i.e. by the Cauchy–Riemann equations). Note that $g(\zeta) = a_m \neq 0$. ■

COROLLARY 41 *Let Ω be a path connected open subset of $\mathbb{C}$, f holomorphic in Ω and $f(\zeta) = 0$ for a point $\zeta \in \Omega$. Then either f vanishes identically in Ω or ζ is an isolated zero of Ω.*

Proof. We apply the previous result. If $f(z) = (z-\zeta)^m g(z)$ with g holomorphic in Ω, $m \in \mathbb{N}$ and $g(\zeta) \neq 0$, then the continuity of g at ζ implies that $g(z) \neq 0$ in some neighbourhood V of ζ. It is then evident that if $f(z) = 0$ and $z \in V$, then $z = \zeta$, i.e. the zero ζ is isolated.

Therefore, if the zero is not isolated, then f vanishes identically in a neighbourhood of ζ. Suppose that $\zeta_1 \in \Omega$ with $f(\zeta_1) \neq 0$. Since Ω is path connected, there is a continuous path

$$\varphi : [0,1] \longrightarrow \Omega$$

with $\varphi(0) = \zeta$ and $\varphi(1) = \zeta_1$. Unfortunately we wish to avoid situations in which φ has an interval of constancy containing 0. Consider first

$$\sigma = \inf\{t; 0 \leq t \leq 1, \varphi(t) \neq \zeta\}.$$

The point $t = 1$ is an element of the set, so the inf is well-defined. By continuity of φ, we see that $0 \leq \sigma < 1$. Now replace φ by $\varphi_1(t) = \varphi((1-t)\sigma + t)$. In this way, we can assume that there a points $t > 0$ arbitrarily close to 0 such that $\varphi(t) \neq \zeta$.

Now consider

$$\tau = \inf\{t; 0 \leq t \leq 1, f(\varphi(t)) \neq 0\}.$$

Since $f(\varphi(1)) \neq 0$, this is the infimum of a nonempty set. By continuity of φ at $t = 0$ and since f vanishes in a neighbourhood of ζ we have $\tau > 0$. For $t < \tau$, $f(\varphi(t)) = 0$ and therefore by continuity $f(\varphi(\tau)) = 0$. By the previous part of the proof, either f vanishes in a neighbourhood of $\varphi(\tau)$ or $\varphi(\tau)$ is an isolated zero of f.

Suppose first that $\varphi(\tau)$ is an isolated zero of f. Then, there exists $\delta > 0$ such that $0 < |z - \varphi(\tau)| < \delta \Longrightarrow f(z) \neq 0$. But, by definition of τ, $f(\varphi(t)) = 0$ for $0 \leq t \leq \tau$ and it follows that

$$0 \leq t \leq \tau \Longrightarrow \text{ either } |\varphi(t) - \varphi(\tau)| = 0 \text{ or } |\varphi(t) - \varphi(\tau)| \geq \delta.$$

But the Intermediate Value Theorem yields that $|\varphi(t) - \varphi(\tau)| = 0$ for all $t \in [0, \tau]$ for else, the value $\frac{1}{2}\delta$ would have to be taken by $t \mapsto |\varphi(t) - \varphi(\tau)|$ on this interval. But this means that φ has $[0, \tau]$ as an interval of constancy, a situation that we were careful to eliminate earlier.

So we are left with the case that f vanishes in a neighbourhood V of $\varphi(\tau)$. But then $f(\varphi(t)) = 0$ for t in a neighbourhood of τ. This contradicts the definition of τ unless $\tau = 1$. Let us understand this in detail. By continuity of φ there exists $\kappa > 0$ such that $|t - \tau| < \kappa$ implies that $\varphi(t) \in V$ and hence that $f(\varphi(t)) = 0$. Since we know already that $f(\varphi(t)) = 0$ for $0 \le t \le \tau$, this yields $f(\varphi(t)) = 0$ for $0 \le t < \tau + \kappa$ provided always that $t \le 1$. This gives a contradiction to the definition of τ unless $\tau = 1$.

But then we have $f(\varphi(\tau)) = 0$ as before which contradicts $f(\zeta_1) \ne 0$. ■

We need to understand this result in context. If Ω is a path connected bounded open subset of $\mathbb{C}$ and f holomorphic in Ω then f can have infinitely many zeros in Ω, but necessarily the zeros accumulate only at the boundary of Ω.

5.2 Bounded Entire Functions

DEFINITION *An* ***entire function*** *is a function that is holomorphic in the entire complex plane.*

THEOREM 42 (LIOUVILLE'S THEOREM) *A bounded entire function is necessarily constant.*

Proof. Suppose that $|f(z)| \le M$ for all z. Then applying Corollary 39 (Cauchy's Estimate) we get

$$|f'(z)| \le \frac{M}{r}$$

for all $r > 0$. It suffices to take r sufficiently large to see that $f'(z) = 0$ for all $z \in \mathbb{C}$. Since f can be recovered from its derivative by means of

$$f(z) = f(0) + \int_\Gamma f'(z) dz$$

where Γ is a C^1 path from 0 to z (for example a line segment), it follows that f is constant.

COROLLARY 43 (FUNDAMENTAL THEOREM OF ALGEBRA) *Let p be a monic polynomial of degree $m \geq 1$, then there exist $\alpha_1, \ldots, \alpha_m$ such that*

$$p(z) = \prod_{k=1}^{m} (z - \alpha_k) \tag{5.1}$$

both in the sense of functions (i.e. that and in the sense that (5.1) holds for all $z \in \mathbb{C}$) and in the sense of polynomials, (i.e. in the ring $\mathbb{C}[z]$).

Proof. The proof is by induction on m. For $m = 1$ the result is evident. Suppose that the result has been proved for monic polynomials of degree $m - 1$ and let p be a monic polynomial of degree m. We claim that p has a zero in $\mathbb{C}$. If not, then consider

$$f(z) = \frac{1}{p(z)}$$

in $\mathbb{C}$. This function is the composition of two holomorphic functions $z \mapsto p(z)$ and $w \mapsto w^{-1}$, the latter being defined on $\mathbb{C} \setminus \{0\}$, where p takes its values. So f is entire. But f is also bounded. to see this, we observe that if $p(z) = \sum_{k=0}^{m} p_k z^k$ with $p_m = 1$ we get

$$\begin{aligned}
|p(z)| &\geq |z|^m - \sum_{k=0}^{m-1} |p_k||z|^k \\
&\geq |z|^m \left(1 - \sum_{k=0}^{m-1} |p_k||z|^{k-m}\right) \\
&\geq \tfrac{1}{2}|z|^m \text{ for } |z| \text{ large,}
\end{aligned}$$

since the quantity in brackets exceeds $\frac{1}{2}$ if $|z|$ is large enough. Since f is continuous, it is necessarily bounded on any bounded set and hence is bounded on the whole of $\mathbb{C}$. So f is constant and $f(0)p(z) = 1$ for all $z \in \mathbb{C}$ which contradicts the fact that p has degree ≥ 1.

Thus, p vanishes somewhere, say at α_m. Then the fact (easily verified) that $z - \alpha_m$ divides $z^k - \alpha_m^k$ for every $k \in \mathbb{Z}^+$ yields that $z - \alpha_m$ divides $p(z) = p(z) - p(\alpha_m)$. So we may write $p(z) = (z - \alpha_m)q(z)$. We see that q is a monic polynomial of degree $m - 1$ and the induction hypothesis finishes the proof. The fact that polynomial functions and polynomials are identical concept on $\mathbb{C}$ is an easy consequence of the standard formula for the vandermonde determinant. We leave the details to the reader. ■

Similar to Liouville's Theorem we have the following proposition.

PROPOSITION 44 *Let f be entire and satisfy $|f(z)| \leq C(1+|z|)^\alpha$ where C and α are absolute constants, $m \in \mathbb{Z}^+$ and $\alpha < m+1$. Then f is a polynomial of degree at most m.*

Proof. Again, Cauchy's Estimate (Corollary 39) gives

$$\begin{aligned} |f^{(m+1)}(z)| &\leq \frac{(m+1)!}{r^{m+1}} \sup_{|\zeta - z| = r} |f(\zeta)| \\ &\leq C\frac{(m+1)!}{r^{m+1}}(1+|z|+r)^{m+\alpha} \end{aligned}$$

Letting $r \longrightarrow \infty$, we get $f^{(m+1)}(z) = 0$ for all $z \in \mathbb{C}$. Integrating up $m+1$ times now shows that f is a polynomial of degree at most m. ■

5.3 The Riemann Sphere and Möbius Transformations

There are various ways of thinking about the Riemann Sphere which will be denoted $\mathbb{C}_\infty$. As a set, it is the abstract union of $\mathbb{C}$ with $\{\infty\}$, the singleton consisting of the point at infinity. From the metric or topological space point of view it is the one point compactification of $\mathbb{C}$. This means that we can put a metric on $\mathbb{C}_\infty$ for which a sequence (z_n) converges to infinity if and only if (z_n) diverges properly to ∞. If you want an explicit metric that embodies this, you could take

$$d(z_1, z_2) = \max\left(\left| \frac{z_1}{1+|z_1|^2} - \frac{z_2}{1+|z_2|^2} \right|, \left| \frac{|z_1|}{1+|z_1|} - \frac{|z_2|}{1+|z_2|} \right| \right)$$

with the specific interpretation

$$d(z, \infty) = d(\infty, z) = \max\left(\left| \frac{z}{1+|z|^2} \right|, \left| \frac{|z|}{1+|z|} - 1 \right| \right)$$

The Riemann Sphere is compact. to see this take a sequence in $\mathbb{C}_\infty$. If the sequence is bounded, then it has a convergent subsequence by the Bolzano–Weierstrass Theorem. If it is unbounded, then it has a subsequence properly divergent to ∞[2].

A more sophisticated way of thinking of $\mathbb{C}_\infty$ is as the complex projective space $\mathbb{CP}_1$, that is the space of one dimensional complex linear subspaces of $\mathbb{C}^2$. For most of those subspaces, we can find a basis vector in the form $(1, z)$, but the subspace $\{0\} \times \mathbb{C}$ is the sole exception to this. If we identify z with $\mathrm{linspan}\{(1, z)\}$, then the point at infinity is identified to the line $\{0\} \times \mathbb{C} = \mathrm{linspan}\{(0, 1)\}$. There is actually nothing special about the one-dimensional subspace $\{0\} \times \mathbb{C}$, to an impartial observer, it looks just like any one-dimensional subspace and consequently, there is nothing special about the point at infinity in $\mathbb{C}_\infty$.

Yet another way of viewing $\mathbb{C}_\infty$ is as a manifold. This is a complicated concept, but one worth exploring a little bit. We start in the real setting. It is clear that one can think for example of the sphere S^2 in $\mathbb{R}^3$ and consider differentiable functions defined just on S^2 (and not on $\mathbb{R}^3 \setminus S^2$). Here, the sphere and its differentiable structure come from the fact that S^2 is embedded in the space $\mathbb{R}^3$. We would like to find a way of describing the differentiable structure of S^2 without reference to this embedding. This is the concept of a manifold. Nicely embedded surfaces are manifolds, but the manifold structure should not reference this embedding.

So, to define a manifold, we first decide upon a dimension d for the manifold (in the case of S^2 this will be 2) and we insist that the manifold M is a metric space. Sometimes additional conditions are imposed already at this point. The manifold M has an atlas which is a collection of charts $\varphi_\alpha : U_\alpha \longrightarrow V_\alpha$ as α runs over an index set I. The subsets U_α are open in M and they cover M, i.e. $\displaystyle\bigcup_{\alpha \in I} U_\alpha = M$. The sets V_α are open subsets of $\mathbb{R}^d$ and the mappings φ_α are continuous bijections with continuous inverses. The analogy with the charts of a real-world atlas is based on the idea that we do not try to understand the world in its entirety. We understand it a bit at a time by examining individual maps (charts) of specific regions. However, it's very important that where two charts overlap, they carry the same information. So there is a compatibility condition which concerns the overlap mapping

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$$

[2]We just showed that $\mathbb{C}_\infty$ is sequentially compact. A well-known theorem tells that every sequentially compact metric space is in fact compact.

This mapping is certainly continuous, but we may impose additional regularity. If we are trying to define a C^k manifold, we will insist that each overlap map is C^k. Observe that the overlap mappings are defined from one open subset of $\mathbb{R}^d$ to another and so questions of differentiability have a perfectly good meaning.

Once this has been done, we can define C^k functions on M. Given say $f : M \longrightarrow \mathbb{R}$, f will be C^k on M if and only if each of the mappings $f \circ \varphi_\beta^{-1} : V_\beta \longrightarrow \mathbb{R}$ is C^k. Here we are using the C^k regularity only as an example. The flavour of the overlap mappings essentially determines the flavour of the manifold. More generally it is possible to define C^k functions from one C^k manifold to another.

This is just a very naive introduction to the subject. In practice there are all sorts of problems that arise[3].

So, to come back to the Riemann Sphere, it is a complex analytic manifold. We can work with just two charts. $U_1 = V_1 = \{z; z \in \mathbb{C}\}$, φ_1 is the identity map. $U_2 = \{z \in \mathbb{C}; z \neq 0\} \cup \{\infty\}$, $V_2 = V_1$,

$$\varphi_2(z) = \begin{cases} z^{-1} & \text{if } z \in \mathbb{C} \setminus \{0\}, \\ 0 & \text{if } z = \infty. \end{cases}$$

Both overlap mappings $\varphi_2 \circ \varphi_1^{-1}$ and $\varphi_1 \circ \varphi_2^{-1}$ are the map $z \mapsto z^{-1}$ on the punctured plane $z \in \mathbb{C}; z \neq 0$ which is holomorphic. Hence with this atlas, $\mathbb{C}_\infty$ is a holomorphic manifold (complex analytic manifold). We have the right to define holomorphic functions on $\mathbb{C}_\infty$.

At first sight, it seems that we have gone to a lot of trouble here for nothing. What are the holomorphic functions from $\mathbb{C}_\infty$ to $\mathbb{C}$? Well they are continuous and $\mathbb{C}_\infty$ is compact, so they are necessarily bounded and they are clearly holomorphic on the subset $\mathbb{C}$ of $\mathbb{C}_\infty$, so according to Liouville's Theorem, they are constant on $\mathbb{C}$ and hence by continuity on $\mathbb{C}_\infty$. However, there might be non-trivial holomorphic mappings from $\mathbb{C}_\infty$ to itself.

The ***Möbius transformations*** on $\mathbb{C}_\infty$ are the transformations induced on $\mathbb{C}_\infty$ by invertible linear transformations of $\mathbb{C}^2$, viewing $\mathbb{C}_\infty$ as projective space in $\mathbb{C}^2$. Let

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

[3] One for example that occurs almost immediately is that completely different atlases can define equivalent structures (two real-world atlases from real-world publishers are supposed to carry the same information, but the selection of charts will be different), so one needs a concept of equivalent atlases.

with $\det(T) = \alpha\delta - \beta\gamma \neq 0$. Then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha z + \beta \\ \gamma z + \delta \end{pmatrix}$$

so that the corresponding transformation of $\mathbb{C}_\infty$ is

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$$

it being understood that the point at infinity maps to $\alpha\gamma^{-1}$ and $-\delta\gamma^{-1}$ maps to the point at infinity. In case $\gamma = 0$ the point at infinity is preserved and the mapping is actually affine (constant plus linear) on $\mathbb{C}$. Note that if $t \in \mathbb{C} \setminus 0$, then the matrix tT yields the same Möbius transformation as T. It is easy to check that Möbius transformations are holomorphic on $\mathbb{C}_\infty$. To check analyticity at ∞, we must check that

$$w \mapsto \frac{\alpha w^{-1} + \beta}{\gamma w^{-1} + \delta} = \frac{\alpha + \beta w}{\gamma + \delta w}$$

is analytic at $w = 0$ and this is OK unless $\gamma = 0$ when we need to check that

$$w \mapsto \left(\frac{\alpha + \beta w}{\gamma + \delta w} \right)^{-1} = \frac{\gamma + \delta w}{\alpha + \beta w} = \frac{\delta w}{\alpha + \beta w}$$

is analytic at $w = 0$. This is good since if $\alpha = 0$ then the determinant condition is violated.

LEMMA 45 *Every Möbius Transformation can be expressed as a composition of translations, dilations and inversions.*

Proof. Throughout the proof, μ denotes a generic non-zero complex number. We have to build up every Möbius Transformation as a composition of Möbius Transformations of one of the following forms:

$$z \mapsto z + \mu, \qquad z \mapsto \mu z, \qquad z \mapsto z^{-1}.$$

Now every Möbius Transformation is given by a non-singular matrix and every non-singular matrix is a product of elementary matrices (remember gaussian reduction). We need therefore, only work with Möbius Transformations that are generated by elementary matrices.

1. $\begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$, $\qquad z \mapsto \mu z$.

2. $\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$, $\qquad z \mapsto \mu^{-1} z$.

3. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\qquad z \mapsto z^{-1}$.

4. $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$, $\qquad z \mapsto z + \mu$.

5. $\begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Cases 1 thru 4 are of the desired type and in case 5 we show how to build this case out of the previous four cases. ■

PROPOSITION 46 *The group of Möbius Transformations is triply transitive on* $\mathbb{C}_\infty$.

Proof. Möbius Transformations clearly form a group since invertible 2×2 matrices do. Let a, b, c be distinct in $\mathbb{C}_\infty$, then consider

$$\varphi(z) = \frac{c(b-a)z + a(c-b)}{(b-a)z + (c-b)}$$

then φ is a genuine Möbius Transformation and φ takes 0, 1 and ∞ to a, b and c respectively. The definition of φ needs modification in certain special cases:

$$\varphi(z) = \begin{cases} \dfrac{cz + c - b}{z} & \text{if } a = \infty, \\ \dfrac{cz - a}{z - 1} & \text{if } b = \infty, \\ \dfrac{(b-a)z + a}{1} & \text{if } c = \infty. \end{cases}$$

Similarly, if a', b', c' are distinct in $\mathbb{C}_\infty$, there exists a Möbius Transformation ψ taking 0, 1 and ∞ to a', b' and c'. It follows that $\psi \circ \varphi^{-1}$ takes a, b and c to a', b' and c' respectively. ■

Actually, the Möbius Transformation that does this is uniquely determined. To see this, suppose that χ_1 and χ_2 both take a, b and c to a', b' and c' respectively. Then $\chi = \chi_1^{-1} \circ \chi_2$ fixes a, b and c. Letting

$$\chi(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

we see that the equation $\chi(z) = z$ has three distinct roots and it follows that the quadratic equation $\gamma z^2 + (\delta - \alpha)z - \beta = 0$ must vanish identically. So $\gamma = \beta = 0$ and $\alpha = \delta$. (Of course, $\alpha\delta - \beta\gamma \neq 0$, so that $\alpha^2 \neq 0$ and $\alpha \neq 0$. We find $\chi(z) = z$ for all $z \in \mathbb{C}_\infty$, or equivalently $\chi_1 = \chi_2$.

We next define the ***cross ratio*** of four distinct complex numbers z_1, z_2, z_3 and z_4 to be

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

The order of the numbers is important. The definition can be extended to the case of $z_j \in \mathbb{C}_\infty$ for $j = 1, 2, 3, 4$ provided $\{z_1, z_2, z_3, z_4\}$ has at least 3 elements but also takes values in $\mathbb{C}_\infty$.

LEMMA 47 *We have* $[\varphi(z_1), \varphi(z_2), \varphi(z_3), \varphi(z_4)] = [z_1, z_2, z_3, z_4]$ *for* φ *a Möbius Transformation.*

Proof. It suffices to check this in the case that φ is a translation, dilation or inversion. In each of these cases, the verification is routine. ■

It is worth observing that $[z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}] = [z_1, z_2, z_3, z_4]$ if σ is a double transposition in S_4. For example, we have

$$[z_2, z_1, z_4, z_3] = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_2 - z_3)(z_4 - z_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = [z_1, z_2, z_3, z_4]$$

Let $\lambda = [z_1, z_2, z_3, z_4]$. Now, it is well known that the set of double transpositions together with the identity permutation form a normal subgroup H of S_4 which is clearly transitive in its action on the 4-tuple (z_1, z_2, z_3, z_4), so each coset of H will contain a permutation that fixes 1 say. There are six such permutations which accounts for all the elements in the coset space (quotient). Now there is a Möbius transform f with $f(z_2) = 0$, $f(z_3) = 1$ and $f(z_4) = \infty$ and since Möbius transforms preserve cross-ratio, we have $[z_1, z_2, z_3, z_4] = \lambda = [\lambda, 0, 1, \infty]$. Thus $f(z_1) = \lambda$. It is therefore sufficient to compute

$$[\lambda, 0, 1, \infty] = \lambda,\ [\lambda, 1, \infty, 0] = (\lambda - 1)/\lambda,\ [\lambda, \infty, 0, 1] = 1/(1 - \lambda),$$

$$[\lambda, 0, \infty, 1] = \lambda/(\lambda - 1),\ [\lambda, \infty, 1, 0] = 1/\lambda,\ [\lambda, 1, 0, \infty] = 1 - \lambda.$$

in order to determine all possible values of $[z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}]$ as σ runs over S_4.

By a ***circle in*** $\mathbb{C}_\infty$, we mean either a circle in $\mathbb{C}$ or a straight line in $\mathbb{C}$ together with the point at infinity. (i.e. we interpret a straight line as a circle that passes thru ∞).

LEMMA 48 *A Möbius transform maps circles to circles.*

Proof. It is easy to see that any circle can be represented by the equation

$$p|z|^2 - 2\Re(\overline{a}z) + q = 0$$

where $a \in \mathbb{C}$, $p, q \in \mathbb{R}$ and $pq < |a|^2$. It is also easy to see that substituting $z = w + \mu$, $z = \mu w$ or $z = w^{-1}$ into such an equation yields an equation of the same type. Since these transformations generate all Möbius transforms, the result follows. ∎

COROLLARY 49 *If z_1, z_2, z_3 and z_4 are distinct points of $\mathbb{C}_\infty$, then $[z_1, z_2, z_3, z_4]$ is real if and only if there is a circle to which all four points belong.*

Proof. We use the fact that the group of Möbius transformations is triply transitive, preserves the cross-ration and maps circles to circles to show that without loss of generality we may take $z_2 = 0$, $z_3 = 1$ and $z_4 = \infty$. In this way, the problem reduces to showing that (for $z \in \mathbb{C} \setminus \{0, 1\}$, $z = [z, 0, 1, \infty]$ is real if and only if z lies on the circle thru 0, 1 and ∞, i.e. the real axis. ∎

Of course, we usually think of a circle as having an inside and an outside. So which is which. If we take z_2, z_3 and z_4 in order and orient the circle S thru z_2, z_3 and z_4 by traversing in that order, then it can be shown that the set $\{z; z \in \mathbb{C}_\infty, \Im[z, z_2, z_3, z_4] > 0\}$ is the connected component of $\mathbb{C}_\infty \setminus S$ "on the left" and $\{z; z \in \mathbb{C}_\infty, \Im[z, z_2, z_3, z_4] < 0\}$ is the connected component of $\mathbb{C}_\infty \setminus S$ "on the right".

5.4 Preservation of Angle

At points where the derivative fails to vanish, holomorphic functions are orientation preserving. In fact, we can actually say much more.

PROPOSITION 50 *Let f be holomorphic in a neighbourhood of a point ζ and suppose that $f'(\zeta) \neq 0$. Then the Jacobian matrix of f is a positive scalar multiple of a rotation matrix.*

Proof. This is an immediate consequence of the Cauchy–Riemann equations. We can write the Jacobian as

$$J = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$$

where $f = u + iv$, u and v are real, $\dfrac{\partial u}{\partial x} = p = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x} = q = -\dfrac{\partial u}{\partial y}$. In case p and q are not both zero, we can write

$$\begin{pmatrix} p & -q \\ q & p \end{pmatrix} = r \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

for $r = \sqrt{p^2 + q^2} > 0$ and suitable real θ. ■

Note that the Jacobian determinant is $r^2 = p^2 + q^2 > 0$, so that locally near ζ, f preserves orientations. We can understand Proposition 50 in the following way. Suppose that $t \mapsto \alpha(t)$ and $t \mapsto \beta(t)$ are two smooth curves thru ζ with $\alpha(0) = \beta(0) = \zeta$. Then $t \mapsto f(\alpha(t))$ and $t \mapsto f(\beta(t))$ are also smooth curves thru $f(\zeta)$ with $f \circ \alpha(0) = f \circ \beta(0) = f(\zeta)$. We have from the chain rule

$$(f \circ \alpha)'(0) = J(\alpha'(0)) \qquad (f \circ \beta)'(0) = J(\beta'(0))$$

The angle between the curves α and β at ζ is just the angle between the vectors $\alpha'(0)$ and $\beta'(0)$ and since J is a positive multiple of a rotation matrix, this is the same as the angle between $(f \circ \alpha)'(0)$ and $(f \circ \beta)'(0)$ which is just the angle between the image curves $f \circ \alpha$ and $f \circ \beta$ at $f(\zeta)$. Thus, where a holomorphic transformation has non-zero derivative, it preserves oriented angle between curves.

Even at a point where the derivative vanishes, we can still say something about angles between curves.

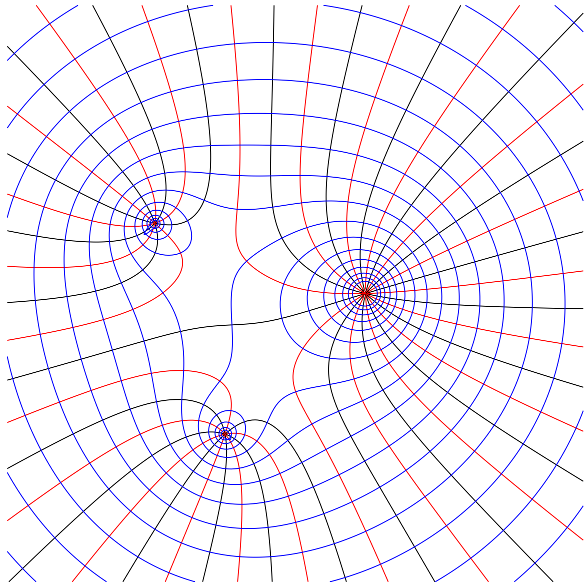

Figure 5.1: Argument – modulus plot.

PROPOSITION 51 *Let Ω be an open subset of $\mathbb{C}$, $\zeta \in \Omega$ and f a holomorphic function on Ω with a zero of order m at ζ. Suppose $1 \le m < \infty$. Then there is a neighbourhood U of ζ and a holomorphic function g defined on U such that $f(z) = \Big(g(z)\Big)^m$ for $z \in U$. Necessarily, g has a single zero at ζ (i.e. $g'(\zeta) \neq 0$).*

Proof. Obviously, if $m = 1$ we can take $U = \Omega$ and $g = f$. By Theorem 40 we can write $f(z) = (z - \zeta)^m h(z)$ with h holomorphic in a neighbourhood of ζ and $h(\zeta) \neq 0$. Now consider an open disk Δ centred at $h(\zeta)$ and of radius $|h(\zeta)|$. Then we can define a holomorphic m^{th} root φ in Δ. In fact, with a little trouble we could even write down a power series expansion for φ centred at $h(\zeta)$ and convergent in Δ. Now set $g(z) = (z - \zeta)\varphi(h(z))$ defined in $h^{-1}(\Delta)$. Then g is holomorphic since it is the product of holomorphic functions ($\varphi \circ h$ being the composition of holomorphic functions). Also we have $(g(z))^m = (z - \zeta)^m(\varphi(h(z)))^m = (z - \zeta)^m h(z) = f(z)$ as required. ∎

What has this to do with angles? Well if f is holomorphic and at a point ζ we have $f'(\zeta) = 0, \ldots, f^{(m-1)}(\zeta) = 0$ but $f^{(m)}(\zeta) \neq 0$, then $z \mapsto f(z) - f(\zeta)$ has a zero of order m at ζ. Therefore, locally about ζ we can write $f(z) =$

$f(\zeta)+(g(z))^m$ where g is holomorphic near ζ and $g'(\zeta) \neq 0$. So, at ζ, g preserves angles and it follows that at ζ the function f multiplies angles by m. To be explicit about this, if we take two curves α and β emanating from ζ then the angle between the curves $f \circ \alpha$ and $f \circ \beta$ will be m times the angle between the curves α and β. It has to be stressed that this occurs only at the point ζ. As soon as you move slightly away from ζ you will necessarily have a non-zero derivative and the usual preservation of angles holds.

Figure 5.1 shows an argument–modulus plot for a function f. To be more explicit, the function shows in the z-plane, curves on which the argument and modulus of $w = f(z)$ are constant. The argument is constant on the red and black curves and the modulus is constant on the blue curves. The increment in the argument is $\pi/6$ (i.e. 30 degrees) and the increments in $\ln(r)$ are also $\pi/6$ (for the curves $|w| = r$). Here are some questions that the reader might like to consider.

- There are three zeros of f in the figure. Where are they?
- One of the zeros is a double zero. Which one? The other two are single zeros. How can you be sure?
- In most places, the blue curves meet the red and black curves at right angles. Why is this?
- The curviquadrilaterals of which most of the figure is composed are approximately square in shape. Why is this?
- How can you be sure that it is not a mixture of zeros and poles that is being displayed. What changes in the figure would you expect if the zero with the largest y coordinate were replaced with a pole?
- Where approximately are the zeros of f'? Two of the regions depicted have eight sides. How do you account for this and what conclusions can you draw? If there were a region with twelve sides, what conclusion would you draw?
- In argument–modulus plots in general, is it possible for a curve of constancy of the modulus to be a smooth figure eight?

THEOREM 52 (OPEN MAPPING THEOREM) *Let Ω be an open path connected subset of $\mathbb{C}$. Let $f : \Omega \longrightarrow \mathbb{C}$ be non-constant and holomorphic. Then for every U open in Ω, $f(U)$ is open in $\mathbb{C}$.*

Proof. It is enough to show that every point ζ of Ω possesses an open neighbourhood V such that $f(V)$ is open. If $f'(\zeta) \neq 0$, the we may apply the Inverse Function Theorem (from MATH 354) to deduce the result. If not, then ζ has finite multiplicity as a zero of f' (since if not then f' would vanish identically on Ω by Corollary 41 and f would be constant on Ω.

So, according to the previous result, we have $f(z) = f(\zeta) + \Big(g(z)\Big)^m$ for z in a neighbourhood of ζ and $g'(\zeta) \neq 0$. It therefore suffices to show that if V is open in $\mathbb{C}$, then $W = \{z^m; z \in V\}$ is also open in $\mathbb{C}$. If $w \in W \setminus \{0\}$, then $w = z^m$ with $z \in V$ and $z \neq 0$ and the result follows as above since $mz^{m-1} \neq 0$. If $0 \in W$, then $0 \in V$ and there exists $\delta > 0$ such that $U(0, \delta) \subseteq V$ whence $U(0, \delta^m) \subseteq W$. ∎

5.5 Morera's Theorem

Morera's Theorem is a converse to Cauchy's Theorem. It's strength lies in the fact that the only regularity imposed on the function is continuity.

THEOREM 53 (MORERA'S THEOREM) *Let f be a continuous function defined on an open subset Ω of $\mathbb{C}$. Suppose that for every triangle T we have $\displaystyle\int_T f(z)dz = 0$. Then f is holomorphic in Ω.*

Proof. Being holomorphic is a local property, so it suffices to prove the result on an open disk whose closure is contained in Ω. Without loss of generality on $\{z; z \in \mathbb{C}, |z| < 1\}$. Define $F(z) = \displaystyle\int_{L(0,z)} f(w)dw$ where $L(a, b)$ denotes the line segment from a to b. Then consider $F(z + h) - F(z)$. Now the three point 0, z and $z + h$ form a triangle T so the hypothesis $\displaystyle\int_T f(z)dz = 0$ can be written

$$F(z+h) - F(z) = \int_{L(0,z+h)} f(w)dw - \int_{L(0,z)} f(w)dw = \int_{L(z,z+h)} f(w)dw.$$

Parametrizing $L(z, z+h)$ by $t \mapsto z + th$ for t running from 0 to 1, we get

$$\int_{L(z,z+h)} f(w)dw = h \int_0^1 f(z+th)dt.$$

It now follows that

$$F(z+h) = F(z) + f(z)h + h \int_0^1 \Big(f(z+th) - f(z)\Big)dt.$$

and

$$\left| h \int_0^1 \Big(f(z+th) - f(z)\Big)dt \right| \le |h| \omega_{f|_K}(h)$$

where $K = \{z; z \in \mathbb{C}, |z| \le 1\}$. Since f is continuous on K it is uniformly continuous and it follows that F has a complex derivative $f(z)$. But f is continuous, so F is holomorphic. But the derivative of a holomorphic function is again holomorphic. ■

Morera's Theorem is not a curiosity, it is a powerful tool. It's main application is the following result which might be less easy to prove by other means.

COROLLARY 54 *Let Ω be an open subset of $\mathbb{C}$. Let (f_n) be a sequence of holomorphic functions converging uniformly on the compacta of Ω to a function f. Then f is holomorphic.*

Proof. Let T be a triangle in Ω, then we will show $\int_T f(z)dz = 0$. Since T will be arbitrary in Ω we can apply Morera's Theorem to obtain the desired conclusion. By Cauchy's Theorem we have $\int_T f_n(z)dz = 0$. We can prove this either from the Green's Theorem version of Cauchy's Theorem, or using the fact that triangles are contractible. The uniform on compacta convergence yields the convergence of the integrals.

$$\begin{aligned} \left| \int_T f(z)dz - \int_T f_n(z)dz \right| &= \left| \int_T (f(z) - f_n(z))dz \right| \\ &\le \sup_{z \in T} |f(z) - f_n(z)| \operatorname{length}(T) \longrightarrow 0 \end{aligned}$$

as $n \longrightarrow \infty$. This gives the desired equality $\int_T f(z)dz = 0$. ■

6

The Maximum Principle

The maximum principle actually applies to harmonic functions rather than analytic functions, so it makes sense to revisit the subject of harmonic functions. It is also very little trouble to tackle this issue in $\mathbb{R}^d$ rather than just in $\mathbb{R}^2$.

6.1 Harmonic Functions Again

We start with some motivational material in two dimensions. Let u be a real-valued harmonic function defined in a neighbourhood of the closed unit disk $\{z \in \mathbb{C}; |z| \leq 1\}$. Then we can find a conjugate harmonic function v defined in a possibly smaller neighbourhood of the closed unit disk. Then, applying the Cauchy Integral formula (Theorem 36) we have

$$\begin{aligned}(u+iv)(z) &= \frac{1}{2\pi}\int_{t=0}^{2\pi} \frac{(u+iv)(e^{it})}{e^{it}-z} e^{it} dt \\ &= \frac{1}{2\pi}\int_{t=0}^{2\pi} (u+iv)(e^{it}) \left(\sum_{n=0}^{\infty} e^{-int} z^n\right) dt\end{aligned}$$

for $|z| < 1$. Now from Cauchy's Theorem we have

$$\frac{1}{2\pi}\int_{t=0}^{2\pi} (u+iv)(e^{it})\, e^{int} dt = \frac{1}{2\pi i}\int (u+iv)(\zeta)\, \zeta^{n-1} d\zeta = 0,$$

where the path for the integral on the right is the unit circle oriented anticlockwise. We can therefore write

$$(u+iv)(z) = \frac{1}{2\pi}\int_{t=0}^{2\pi} (u+iv)(e^{it}) \left(\sum_{n=0}^{\infty} e^{-int} z^n + \sum_{n=1}^{\infty} e^{int} \overline{z}^n\right) dt$$

again for $|z| < 1$. Computing the geometric sums in the brackets and combining, we get

$$(u+iv)(z) = \frac{1}{2\pi}\int_{t=0}^{2\pi}(u+iv)(e^{it})\frac{1-|z|^2}{|z-e^{it}|^2}dt.$$

But the function $\dfrac{1-|z|^2}{|z-e^{it}|^2}$ is positive and taking real parts we get

$$u(z) = \frac{1}{2\pi}\int_{t=0}^{2\pi}u(e^{it})\frac{1-|z|^2}{|z-e^{it}|^2}dt$$

which is the Poisson Integral formula for a harmonic function. The case $z=0$ is particularly important

$$u(0) = \frac{1}{2\pi}\int_{t=0}^{2\pi}u(e^{it})dt$$

and tells that the average value of a harmonic function on a circle is the value at the centre of the circle.

6.2 Digression – Poisson Integrals in $\mathbb{R}^n$

This whole theory works out almost as well in $\mathbb{R}^d$. So imagine that a real-valued function f is continuous on the unit sphere $S^{d-1} = \{y \in \mathbb{R}^d; |y| = 1\}$ in $\mathbb{R}^d$. We will form the Poisson Integral of f,

$$F(x) = \frac{1}{A_{d-1}}\int f(y)\frac{1-|x|^2}{|x-y|^d}d\sigma(y) \tag{6.1}$$

for $|x| < 1$ and where σ is the $d-1$ dimensional area measure on S^{d-1} and A_{d-1} denotes the $d-1$ dimensional area of S^{d-1}.

Let now $y \in S^{d-1}$ be fixed and let $p(x) = 1-|x|^2$ and $q(x) = |x-y|^{-d}$. We find $\triangle(pq) = p\triangle q + 2\nabla p\cdot\nabla q + q\triangle p$, using the formula $\triangle v = \dfrac{\partial^2 v}{\partial r^2} + \dfrac{d-1}{r}\dfrac{\partial v}{\partial r}$ for radial functions. This gives $\triangle(q) = d(d+1)|x-y|^{-d-2} - d(d-1)|x-y|^{-d-2} = 2d|x-y|^{-d-2}$ and

$$\begin{aligned}
\triangle(pq) &= 2d(1-|x|^2)|x-y|^{-d-2} + 4d|x-y|^{-d-2}x\cdot(x-y) - 2d|x-y|^{-d} \\
&= 2d|x-y|^{-d-2}\Big(1-|x|^2+2|x|^2-2x\cdot y-|x|^2+2x\cdot y-|y|^2\Big) \\
&= 2d|x-y|^{-d-2}\Big(1-|y|^2\Big) = 0,
\end{aligned}$$

since $|y| = 1$. Hence, differentiating twice (i.e. applying the Laplace operator) under the integral sign in (6.1), we get $\triangle F = 0$. The differentiation under the integral sign is valid because so long as x is kept in a ball $|x| \leq r$ with $0 < r < 1$ and $|y| = 1$, the function $x \mapsto \dfrac{1 - |x|^2}{|x - y|^d}$ is infinitely differentiable with derivatives of all orders being bounded and continuous and the integral over the sphere will somehow be built up out of iterated integrals involving $d - 1$ one-dimensional Riemann integrals. It will therefore suffice to apply the regular differentiation under the integral sign theorem, multiple times.

Our objective is

THEOREM 55 *For f a continuous function on the unit sphere S^{d-1} we define $F(x)$ by* (6.1) *if $|x| < 1$ and $F(x) = f(x)$ if $|x| = 1$, thereby obtaining a function on the closed unit ball. We claim that F is continuous on the closed unit ball and harmonic on the open unit ball. In the parlance of PDE, F solves the* ***Dirichlet Problem*** *on the unit ball.*

Proof. In case $d = 1$ we proceed by direct calculation to verify that

$$F(x) = \frac{1}{2}\Big(f(1)(1 + x) + f(-1)(1 - x)\Big)$$

which is easily seen to satisfy the requirements of the theorem. Hence we may always assume that $d \geq 2$.

Now consider the case where $f \equiv 1$, we have

$$G(x) = \frac{1}{A_{d-1}} \int \frac{1 - |x|^2}{|x - y|^d} d\sigma(y)$$

is harmonic in the open unit ball and it is clear from the definition that $G(x)$ depends only on $|x|$. This is ultimately a consequence of the fact that the area measure σ on S^{d-1} is rotationally invariant. But, we can solve for radial harmonic functions and we obtain

$$G(x) = \begin{cases} A + B\ln(|x|) & \text{if } d = 2, \\ A + B|x|^{-d+2} & \text{if } d \geq 3. \end{cases}$$

Since G is clearly bounded, we deduce in either case that $B = 0$ and therefore G is constant. But $G(0) = 1$, so we find $G(x) = 1$ for all x with $|x| < 1$.

To complete the proof of the theorem, we proceed to cut F onto spherical shells. The function f_r will be essentially the restriction of F to a sphere centred at the origin of radius r, but parametrized on the unit sphere, namely

$$\begin{aligned} f_r(z) = F(rz) &= \frac{1}{A_{d-1}} \int f(y) \frac{1-r^2}{|rz-y|^d} d\sigma(y) \text{ for } |z| = 1 \\ &= \int P_r(z,y) f(y) d\sigma(y) \end{aligned}$$

where $P_r(z,y) = \dfrac{1}{A_{d-1}} \dfrac{1-r^2}{|rz-y|^d}$. We claim that P_r is a summability kernel, explicitly

- $P_r(z,y) \geq 0$, for all $z, y \in S^{d-1}$ and $0 \leq r < 1$.
- $\displaystyle\int P_r(z,y) d\sigma(y) = 1$ for all $z \in S^{d-1}$ and $0 \leq r < 1$.
- $\displaystyle\sup_{z \in S^{d-1}} \int_{|z-y|>\delta} P_r(z,y) d\sigma(y) \underset{r \to 1-}{\longrightarrow} 0$ for all $\delta > 0$.

The Summability Kernel Theorem then shows that $f_r \longrightarrow f$ uniformly on S^{d-1} as $r \to 1-$. A moment's thought shows that this implies the continuity of F on the closed unit ball.

The first and second conditions for a summability kernel have been shown. It remains to verify the third. We claim that

$$|rz - y| \geq \tfrac{1}{2}|z - y|.$$

To see this, it suffices to show

$$4(r^2|z|^2 - 2rz \cdot y + |y|^2) \geq |z|^2 - 2z \cdot y + |y|^2$$

or equivalently

$$4r^2 + (2 - 8r) z \cdot y + 2 \geq 0.$$

When $0 \leq r \leq \frac{1}{4}$ the worst case is when $z \cdot y = -1$. It boils down to $4r(r+2) \geq 0$ and when $\frac{1}{4} \leq r < 1$, the worst case is when $z \cdot y = 1$ leading to $4(1-r)^2 \geq 0$. The claim is established. Therefore

$$\int_{|z-y|>\delta} P_r(z,y) d\sigma(y) \leq \frac{1}{A_{d-1}} \int_{S^{d-1}} \frac{(1-r^2)2^d}{\delta^d} d\sigma(y) = 2^d \delta^{-d} (1 - r^2).$$

The proof is complete. ■

Let us now back up and establish the result on summability kernels. We will state this in terms of a sequence of kernels φ_n on a compact metric space X and some "measure" μ on X (this concept is made precise in MATH 355). We first assume that φ_n is sufficiently regular on $X \times X$. Continuity on $X \times X$ is sufficient, but there are cases where one is interested in kernels that are not continuous. Then come the three defining properties

- $\varphi_n(x, y) \geq 0$ for all $x, y \in X$ and $n \in \mathbb{N}$.
- $\displaystyle\int_X \varphi_n(x, y) d\mu(y) = 1$ for all $x \in X$ and $n \in \mathbb{N}$.
- For all $\delta > 0$ we have $\displaystyle\sup_{x \in X} \int_{d(x,y)>\delta} \varphi_n(x, y) d\mu(y) \underset{n\to\infty}{\longrightarrow} 0$

THEOREM 56 (SUMMABILITY KERNEL THEOREM) *Let $(\varphi_n)_{n=1}^{\infty}$ be a summability kernel as above. Let f be a continuous real-valued function on X (complex-valued will also work). Then*

$$f_n(x) = \int \varphi_n(x, y) f(y) d\mu(y) \longrightarrow f(x)$$

uniformly on X as $n \to \infty$.

Proof. We have by the second condition

$$f_n(x) - f(x) = \int \varphi_n(x, y)\Big(f(y) - f(x)\Big)d\mu(y)$$

and by the first

$$|f_n(x) - f(x)| \leq \int \varphi_n(x, y)\Big|f(y) - f(x)\Big|d\mu(y)$$

Let $\epsilon > 0$. Then, since f is uniformly continuous (it is continuous on a compact space), we can find $\delta > 0$ such that $\omega_f(\delta) < \frac{1}{2}\epsilon$. So

$|f_n(x) - f(x)|$

$$
\begin{aligned}
&\le \int_{d(x,y)\le\delta} \varphi_n(x,y)\Big|f(y)-f(x)\Big|d\mu(y) + \int_{d(x,y)>\delta} \varphi_n(x,y)\Big|f(y)-f(x)\Big|d\mu(y)\\
&\le \int_X \varphi_n(x,y)\omega_f(\delta)d\mu(y) + 2\sup_{x\in X}|f(x)|\int_{d(x,y)>\delta}\varphi_n(x,y)d\mu(y)\\
&\le \frac{1}{2}\epsilon + 2\sup_{x\in X}|f(x)|\int_{d(x,y)>\delta}\varphi_n(x,y)d\mu(y) < \epsilon
\end{aligned}
$$

provided that n is large enough by the third condition. ■

6.3 Maximum Principles for Harmonic Functions

PROPOSITION 57 (FIRST MAXIMUM PRINCIPLE) *Let Ω be a bounded open subset of $\mathbb{R}^d$. Suppose that f is continuous real-valued on $\mathrm{cl}(\Omega)$ and harmonic in Ω. Then*

$$\sup_{x\in \mathrm{cl}(\Omega)} f(x) \le \sup_{x\in\partial\Omega} f(x)$$

Proof. Suppose not. Then there is a point $z\in\Omega$ such that

$$f(z) > \sup_{x\in\partial\Omega} f(x).$$

Now $\partial\Omega$ is bounded, so there exists $\epsilon>0$ such that

$$\epsilon \sup_{x\in\partial\Omega}|x-z|^2 < f(z) - \sup_{x\in\partial\Omega} f(x).$$

Now define $g(x) = f(x) + \epsilon|x-z|^2$. We have

$$g(z) = f(z) > \sup_{x\in\partial\Omega} f(x) + \epsilon\sup_{x\in\partial\Omega}|x-z|^2 \ge \sup_{x\in\partial\Omega} g(x) \tag{6.2}$$

Now, $\mathrm{cl}(\Omega)$ is compact and g is continuous on $\mathrm{cl}(\Omega)$, so it attains its maximum value at a point $y\in\mathrm{cl}(\Omega)$. But because of (6.2), we see that the maximum cannot be taken on $\partial\Omega$. It follows that $y\in\mathrm{cl}(\Omega)\setminus\partial\Omega=\Omega$. It follows from basic calculus that the Hessian of g at y is negative semi-definite. In particular the trace of the Hessian is ≤ 0. But the trace of the Hessian is just the Laplacian, so

$$0 \ge \triangle g(y) = \triangle f(y) + 2\epsilon d = 2\epsilon d > 0$$

a contradiction. ■

COROLLARY 58 *Let Ω be a bounded open subset of $\mathbb{R}^d$. Suppose that f is continuous complex-valued on* $\mathrm{cl}(\Omega)$ *and harmonic in* Ω*. Then*

$$\sup_{x \in \mathrm{cl}(\Omega)} |f(x)| \le \sup_{x \in \partial\Omega} |f(x)|$$

Proof. Suppose not. Then there is a point $z \in \Omega$ such that

$$|f(z)| > \sup_{x \in \partial\Omega} |f(x)|.$$

Let ω be the complex sign of $f(z)$. Then let $g(x) = \Re\overline{\omega}f(x)$. Then g is continuous real-valued on $\mathrm{cl}(\Omega)$ and harmonic in Ω. But

$$g(z) = |f(z)| > \sup_{x \in \partial\Omega} |f(x)| \ge \sup_{x \in \partial\Omega} g(x)$$

contradicting Proposition 57. ■

A further corollary is now given.

COROLLARY 59 *Let G be a function continuous on the closed unit ball and harmonic on the open unit ball. Then*

$$G(x) = \frac{1}{A_{d-1}} \int G(y) \frac{1 - |x|^2}{|x - y|^d} d\sigma(y). \tag{6.3}$$

for $|x| < 1$. In particular, putting $x = 0$, we have

$$G(0) = \frac{1}{A_{d-1}} \int G(y) d\sigma(y),$$

the mean-value property.

Proof. Let $F(x)$ be the right-hand side of (6.3) for $|x| < 1$ and $F(x) = G(x)$ for $|x| = 1$. Then according to Theorem 55, F is continuous on the closed unit ball and harmonic on the open unit ball. We consider $H(x) = F(x) - G(x)$. Then by Corollary 58

$$\sup_{|x| \le 1} |H(x)| \le \sup_{|x| = 1} |H(x)| = 0.$$

So, H vanishes identically and we have our result. ■

Obviously, this last corollary can be scaled to any ball.

PROPOSITION 60 (SECOND MAXIMUM PRINCIPLE) *Let Ω be a open connected subset of $\mathbb{R}^d$. Suppose that f is real-valued harmonic in Ω and that f attains its maximum in Ω. Then f is constant in Ω.*

Proof. Let the maximum value be M. Let $Z = \{x \in \Omega; f(x) = M\}$. Then Z is a non-empty subset of Ω which is relatively closed in Ω. It is the inverse image of a singleton by a continuous function and singletons are necessarily closed. We will show that Z is open. Let $x \in Z$ and choose $r > 0$ such that $U(x, r) \subseteq \Omega$. Consider $0 < t < r$ and according to the mean-value principle scaled to a ball of radius t about x that

$$M = f(x) = \frac{1}{A_{d-1}} \int_{S^{d-1}} f(x + ty) d\sigma(y). \tag{6.4}$$

Now clearly $f(x + ty) \leq M$. Suppose that for some z with $|z| = 1$ we have $f(x + tz) < M$, then, using the continuity of f, there is an $\epsilon > 0$ and an open neighbourhood V of z in S^{d-1} such that $f(x + ty) < M - \epsilon$ for all y in V. But now

$$\begin{aligned}
\int_{S^{d-1}} f(x+ty)d\sigma(y) &\leq \int_{S^{d-1}\setminus V} f(x+ty)d\sigma(y) + \int_V f(x+ty)d\sigma(y) \\
&\leq M \int_{S^{d-1}\setminus V} d\sigma(y) + (M - \epsilon) \int_V d\sigma(y) \\
&= M \int_{S^{d-1}} d\sigma(y) - \epsilon \int_V d\sigma(y) < M A_{d-1}
\end{aligned}$$

leading to a contradiction with (6.4). Hence $f(z) = M$ on $U(x, r)$. It follows that Z is open relative to Ω. But Ω is a connected set and it follows that $Z = \Omega$. ■

COROLLARY 61 *Let Ω be a open connected subset of $\mathbb{R}^d$. Suppose that f is complex-valued harmonic in Ω and that $|f|$ attains its maximum in Ω. Then f is constant in Ω.*

Proof. Let $|f|$ attain its maximum M at $z \in \Omega$. Let ω be the complex sign of $f(z)$. Then let $g(x) = \Re\overline{\omega} f(x)$. Then g is real-valued harmonic in Ω, $g(z) = M$ and $g(x) \leq M$ for all $x \in \Omega$. So, according to Proposition 60, g is constant in Ω. This says that $\Re\overline{\omega} f(x) = M$ for all $x \in \Omega$ and it follows that $|f(x)| \geq M$ for all

$x \in \Omega$. But $|f(x)| \leq M$ for all $x \in \Omega$ so $|f(x)| = M$ for all $x \in \Omega$. Combining this with $\Re\overline{\omega}f(x) = M$, we see that $f(x) = M\omega$ for all $x \in \Omega$. ■

Since in $d = 2$, holomorphic functions are examples of complex-valued harmonic function, the results above for complex-valued harmonic functions also holds in that context. We do have the following result.

PROPOSITION 62 *Let f be holomorphic in a connected open set Ω of $\mathbb{C}$ and suppose that $|f|$ attains its minimum value. Then either f is constant, or the minimum value is zero.*

Proof. Suppose that the minimum value is not zero. Then $|f(z)| > 0$ for all $z \in \Omega$. Hence $g(z) = (f(z))^{-1}$ is holomorphic in Ω and attains its maximum value. By Corollary 61, g is constant in Ω and hence, so is f.

We can now answer the question posed on page 67 concerning holomorphic functions with constant modulus on a figure eight curve. Certainly the constant modulus cannot be zero, for the zero set of a holomorphic function either consists of isolated points unless the function is locally zero. Clearly the function is not locally constant and by compactness it must attain its minimum modulus in each of the closed loops of the figure eight. So, by Proposition 62, the function must vanish somewhere inside each of the closed loops. So, let us try to construct such a function and see what happens. We take $f(z) = 1 - z^2$ with zeros at ± 1. At $z = 0$ the function f takes the value 1. Where do we have $|f(z)| = 1$. We will use polar coordinates. The equation $|1-z^2| = 1$ becomes $(1-r^2\cos(2\theta))^2 + r^4(\sin(2\theta))^2 = 1$ which simplifies to $r^4 - 2r^2\cos(2\theta) = 0$ or, since the origin is on the curve anyway, $r^2 = 2\cos(2\theta)$. This curve is called a lemniscate and has exactly the form of a figure eight. So the answer to the question is yes.

6.4 The Phragmen–Lindelöf Method

The first result is an extension of the maximum modulus principle to a strip. Proposition 57 does not apply because a strip is unbounded.

PROPOSITION 63 *Let $\Omega = \{x + iy; x, y \text{ real}, |y| < \frac{\pi}{2}\}$, f be continuous on the closure of Ω, holomorphic in Ω and satisfy $|f(z)| \leq 1$ for $z \in \partial\Omega$. Further, suppose that there are constants $a < \infty$ and $b < 1$ such that*

$$|f(z)| \leq \exp(ae^{b|\Re z|}) \quad \text{for all } z \in \Omega. \tag{6.5}$$

Then $|f(z)| \leq 1$ for $z \in \Omega$.

The growth condition (6.5) is necessary. Consider for example $f(z) = \exp(e^z)$. Roughly speaking the proposition says that the growth has to be really wild before the maximum modulus principle will fail on the strip.

Proof. Choose c with $b < c < 1$. let $\epsilon > 0$ be arbitrary, but fixed for the moment. Let

$$h_\epsilon(z) = \exp(-\epsilon \cosh(cz))$$

an entire function. Then for $z = x + iy \in \Omega$, with x, y real,

$$\Re \cosh(cz) = \cosh(cx)\cos(cy) \geq \delta \cosh(cx)$$

where $\delta = \cos(c\pi/2) > 0$. Thus

$$|f(z)h_\epsilon(z)| \leq \exp(ae^{b|x|} - \epsilon\delta \cosh(cx)) \longrightarrow 0$$

as $|x| \to \infty$ since $b < c$. On $\partial\Omega$

$$|f(z)h_\epsilon(z)| \leq \exp(-\epsilon\delta \cosh(cx)) \leq 1.$$

We now apply Proposition 57 on the open subset $\Omega_R = \{x + iy; x, y \text{ real }, |x| < R, |y| < \frac{\pi}{2}\}$ where R is sufficiently large. The result is that

$$|f(z)h_\epsilon(z)| \leq 1 \ \text{ for all } z \in \Omega_R.$$

Letting R increase to infinity, we now obtain

$$|f(z)h_\epsilon(z)| \leq 1 \ \text{ for all } z \in \Omega,$$

or equivalently $|f(z)| \leq \left|e^{\epsilon \cosh(z)}\right|$. Letting ϵ decrease to zero now gives the desired result. ∎

The next step in this saga is the following theorem which turns out to play a crucial role in functional analysis. To go into the details of why this result is so important is unfortunately beyond the scope of this course.

THEOREM 64 (THE THREE LINES THEOREM) *Let $\Omega = \{x + iy; x, y$ real $, 0 < x < 1\}$, f be continuous on the closure of Ω, holomorphic in Ω. Suppose that there are positive constants M_0, M_1, $a < \infty$ and $b < \pi$ such that*

$$|f(x+iy)| \leq \begin{cases} \exp(ae^{b|y|}) & \text{for } 0 < x < 1, y \in \mathbb{R}, \\ M_0 & \text{for } x = 0, y \in \mathbb{R}, \\ M_1 & \text{for } x = 1, y \in \mathbb{R}. \end{cases}$$

Then $|f(x+iy)| \leq M_0^{1-x} M_1^x$ for $0 < x < 1, y \in \mathbb{R}$.

Proof. We start by adapting the previous result to the strip involved here. We then apply that result to the function

$$g(z) = M_0^{-1+z} M_1^{-z} f(z).$$

The desired conclusion follows. ■

7

Isolated Singularities and Residues

Morally speaking, a singularity is a place where a function has bad behaviour or is not properly defined, with the understanding that the function has good behaviour nearby.

DEFINITION *Let $\zeta \in \Omega$ where Ω is open in $\mathbb{C}$. Then a function (defined on Ω) which is holomorphic in a punctured neighbourhood $V' = \{z \in \mathbb{C}; 0 < |z-\zeta| < r\}$ of ζ or radius $r > 0$ is said to have an* ***isolated singularity*** *at ζ. Here we are assuming that $V = \{z \in \mathbb{C}; |z-\zeta| < r\} \subseteq \Omega$*

There is a classification of isolated singularities as follows — removable singularities (i.e. singularities that aren't really there at all), poles and essential singularities (i.e. everything else). Here are the definitions.

DEFINITION *Let $\zeta \in \Omega$ where Ω is open in $\mathbb{C}$. Let f be a function defined on Ω which is holomorphic in V'. Then*

- *ζ is a* ***removable singularity*** *, if there is a number $a \in \mathbb{C}$ such that the function $\tilde{f}$ defined by*

$$\tilde{f}(z) = \begin{cases} a & \text{if } z = \zeta, \\ f(z) & \text{if } z \in V'. \end{cases}$$

  is holomorphic in V.

- *ζ is a* ***pole*** *if the function $\tilde{f}$ defined by*

$$\tilde{f}(z) = \begin{cases} \infty & \text{if } z = \zeta, \\ f(z) & \text{if } z \in V'. \end{cases}$$

is holomorphic as a map from V to $\mathbb{C}_\infty$.

- *ζ is an* ***essential singularity*** *if it does not fit into either of the two above cases.*

THEOREM 65 *Let f be a function bounded and holomorphic in V'. Then f has a removable singularity at ζ.*

In fact, boundedness is too strong, one can get away with replacing the boundedness with the condition

$$\lim_{z\to\zeta} |z-\zeta||f(z)| = 0.$$

Proof. The idea is to define g in V by

$$g(z) = \begin{cases} 0 & \text{if } z = \zeta, \\ (z-\zeta)^2 f(z) & \text{if } z \in V'. \end{cases}$$

Then clearly g has a complex derivative $g'(z) = (z-\zeta)^2 f'(z) + 2(z-\zeta)f(z)$ in V'. Also

$$\frac{g(z)-g(\zeta)}{z-\zeta} = (z-\zeta)f(z) \longrightarrow 0$$

as $z \to \zeta$. Hence $g'(\zeta)$ exists and equals zero. We claim that g is holomorphic in V. It remains only to check that g' is continuous in V. This is obvious, except at ζ. Let $\epsilon > 0$. By hypothesis, there exists $\delta > 0$ such that

$$0 < |w-\zeta| < \delta \Longrightarrow |w-\zeta||f(w)| < \epsilon$$

Now choose z with $0 < |z-\zeta| < \frac{1}{2}\delta$. We apply the Cauchy estimate on the disk $\{w \in \mathbb{C}; |w-z| \le \frac{1}{2}|z-\zeta|\} \subset V'$ to get

$$\begin{aligned} |g'(z)| &\le \frac{2}{|z-\zeta|} \sup_{|w-z|=\frac{1}{2}|z-\zeta|} |g(w)| \\ &\le \frac{2}{|z-\zeta|} \sup_{|w-z|=\frac{1}{2}|z-\zeta|} |w-\zeta|^2 |f(w)| \\ &\le 3 \sup_{|w-z|=\frac{1}{2}|z-\zeta|} |w-\zeta||f(w)| \\ &\le 3\epsilon \end{aligned}$$

since $0 < \frac{1}{2}|z-\zeta| \leq |w-\zeta| \leq \frac{3}{2}|z-\zeta| \leq \frac{3}{4}\delta$. it follows that g' is continuous at ζ and completes the claim.

But now we may expand g as a power series $g(z) = \sum_{n=0}^{\infty} a_n (z-\zeta)^n$. Since $g(\zeta) = g'(\zeta) = 0$, it follows that $a_0 = a_1 = 0$ and therefore

$$f(z) = \sum_{n=0}^{\infty} a_{n+2}(z-\zeta)^n$$

for $z \in V'$. ■

COROLLARY 66 *If f is holomorphic in V' and tends properly to ∞ at ζ, then f has a pole at ζ.*

Proof. Consider $g(z) = \dfrac{1}{f(z)}$ defined on $W' = \{z \in V'; f(z) \neq 0\}$. Then g is holomorphic on W' and tends to zero as $z \longrightarrow \zeta$. Hence g extends to a holomorphic function $\tilde{g}$ on $W = \{\zeta\} \cup \{z \in V'; f(z) \neq 0\}$ which is a neighbourhood of ζ. It follows that f has a pole at ζ. ■

Since $\tilde{g}$ has a zero at ζ and $\tilde{g}$ is not identically zero in a neighbourhood of ζ, it follows that $\tilde{g}$ has a zero of order m at ζ where m is an integer $m \geq 1$. This number is called the ***order of the pole*** . In case $m = 1$, we say that ζ is a ***simple pole*** or ***single pole*** . If $m = 2$, we say that ζ is a ***double pole*** . If f has a pole of order m at ζ, then $z \mapsto (z-\zeta)^m f(z)$ has a removable singularity at ζ. Also, if f has a pole of order m at ζ we can express $f(z)$ in terms of an expansion

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-\zeta)^n$$

valid in $0 < |z-\zeta| < \rho$ for some $\rho > 0$. Such an expansion is called a ***Laurent expansion*** . It will turn out that if f has an essential singularity at ζ, then it has a Laurent expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\zeta)^n$$

but this expansion does not terminate as we progress backwards through the negative powers.

EXAMPLE Let $f(z) = \dfrac{\sin(z)}{z}$. Then officially, f is undefined at $z = 0$, but is holomorphic on $\mathbb{C} \setminus \{0\}$. It is easy to see that 0 is a removable singularity and indeed, we have

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} z^{2k}$$

with infinite radius. □

EXAMPLE Let $f(z) = \dfrac{z}{\exp(z) - 1 - z}$. Then officially, f is defined and holomorphic on $\{z \in \mathbb{C}; \exp(z) - 1 - z \neq 0\}$. At every point $\zeta \neq 0$ where $exp(\zeta) - 1 - \zeta = 0$, there is a simple pole at ζ. This is because $z \mapsto \exp(z) - 1 - z$ has a simple zero at ζ. We can tell that the zero is simple, because if not then both $\exp(z) - 1 - z$ and its derivative $\exp(z) - 1$ both vanish at $z = \zeta$ and this can only happen if $\zeta = 0$. At $\zeta = 0$, we see that $z \mapsto \exp(z) - 1 - z$ has a double zero and $z \mapsto z$ has a simple zero. Hence f also has a simple pole at $z = 0$. □

EXAMPLE Let $f(z) = \exp\left(z + \dfrac{1}{z}\right)$, a holomorphic function on $\mathbb{C} \setminus \{0\}$. It is fairly clear that f has an essential singularity at $z = 0$. To prove this rigorously, let y be real with $|y|$ large and solve the equation $z + z^{-1} = iy$, or $z^2 - iyz + 1 = 0$. The sum of the two roots is iy so at least one of them has absolute value $\geq \frac{1}{2}|y|$. The product of the two roots is 1, so the other root has absolute value $\leq 2|y|^{-1}$. As iy tends to ∞, the smaller root tends to zero and the value of f at the smaller root is $\cos(y) + i\sin(y)$ which does not converge to anything, nor does it properly diverge to ∞. □

We can also discuss the concept of singularities at ∞. We are thinking of the Riemann Sphere and using the chart $z \mapsto z^{-1}$ which acts as the chart mapping near ∞. So, f has a removable singularity (respectively pole) at ∞ if the function $z \mapsto f(z^{-1})$ has a removable singularity (respectively pole) at 0. We see that such a function has an expansion

$$f(z) = \sum_{n=-\infty}^{m} a_n z^n$$

with m an integer $m \leq 0$ in the case of a removable singularity and $m \geq 1$ in the case of a pole (in which case, m is the order of the pole).

A function holomorphic on $\mathbb{C}_\infty$ except at finitely many points (if the singularities of f are isolated, this will force them to be finite in number by compactness) where there are poles is necessarily a rational function (i.e. quotient of polynomials). To see this, let f be such a function. Now construct a polynomial q whose zeros match the poles of f in $\mathbb{C}$. If a pole of f has order m, then we insist that the corresponding zero of q has order m. Then $p = qf$ has no poles in $\mathbb{C}$ (they have become removable singularities). At infinity, it has a pole of order m where m is the order of the pole of f at ∞ plus the degree of q. It now follows that there is a constant C such that $|p(z)| \le C(1 + |z|)^m$. Now apply Proposition 44 to show that p is a polynomial.

7.1 Laurent Expansions

Laurent expansions are more general than the expansions introduced earlier. They apply to holomorphic functions defined in annuli. For convenience, we will take our annuli centred at 0.

THEOREM 67 *Let $0 \le r < R \le \infty$ and let Ω be the annulus defined by $r < |z| < R$. If f is holomorphic in Ω, then we may find $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}$ such that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{7.1}$$

where convergence is uniform on the compacta of Ω. Furthermore, we have

$$a_n = \frac{1}{2\pi\rho^n} \int_{\theta=0}^{2\pi} f(\rho e^{i\theta}) e^{-in\theta} d\theta \tag{7.2}$$

for all ρ with $r < \rho < R$.

Proof. Let z_0 be fixed. Choose r_1 and R_1 such that $r < r_1 < |z_0| < R_1 < R$. We start with a small circular contour Γ of radius s oriented anticlockwise and centred at z_0 and lying in Ω. By the Cauchy Integral Formula, we have

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

for all z such that $|z - z_0| < s$. Now consider the following contour, Γ_1. Let $z = \rho e^{i\phi}$. We start at $-R_1 e^{i\phi}$ make an anticlockwise loop around $|z| = R_1$,

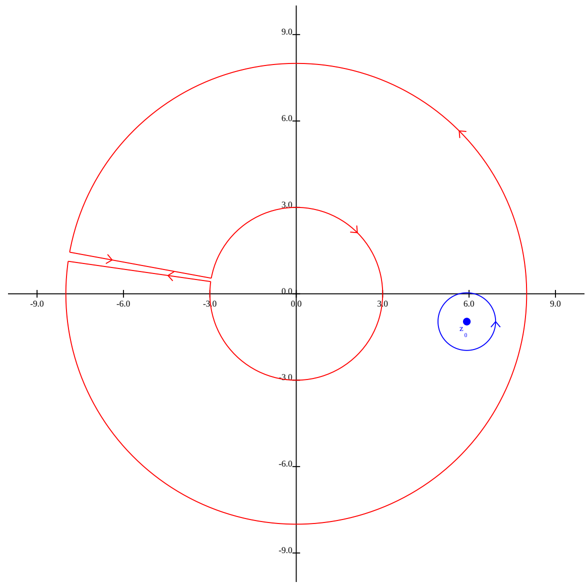

Figure 7.1: Contour used in the proof of Theorem 67.

returning to $-R_1 e^{i\phi}$, then along the straight line path to $-r_1 e^{i\phi}$ then a clockwise loop around $|z| = r_1$, returning to $-r_1 e^{i\phi}$ and finally back along the straight line path to $-R_1 e^{i\phi}$. It is fairly clear that Γ and Γ_1 are homotopic in $\Omega \setminus \{z; |z - z_0| \leq \frac{1}{2}s\}$. So again for $|z - z_0| < \frac{1}{2}s$, we get

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \frac{1}{2\pi} \left(\int_0^{2\pi} \frac{R_1 e^{i\theta}}{R_1 e^{i\theta} - z} f(R_1 e^{i\theta})\, d\theta - \int_0^{2\pi} \frac{r_1 e^{i\theta}}{r_1 e^{i\theta} - z} f(r_1 e^{i\theta})\, d\theta \right)
\end{aligned}
$$

since the integrals over the straight line portions of Γ_1 cancel

$$
\begin{aligned}
&= \frac{1}{2\pi} \left(\int_0^{2\pi} \frac{f(R_1 e^{i\theta})}{1 - R_1^{-1} e^{-i\theta} z}\, d\theta + \int_0^{2\pi} \frac{z^{-1} r_1 e^{i\theta} f(r_1 e^{i\theta})}{1 - z^{-1} r_1 e^{i\theta}}\, d\theta \right) \\
&= \sum_{n=-\infty}^{\infty} a_n z^n
\end{aligned}
$$

where

$$
a_n = \frac{1}{2\pi R_1^n} \int_{\theta=0}^{2\pi} f(R_1 e^{i\theta}) e^{-in\theta} d\theta
$$

for $n \geq 0$ and

$$a_n = \frac{1}{2\pi r_1^n} \int_{\theta=0}^{2\pi} f(r_1 e^{i\theta}) e^{-in\theta} d\theta$$

for $n < 0$. If M is an upper bound for $|f|$ on $r_1 \leq |z| \leq R_1$, then we have $|a_n| \leq R_1^{-n} M$ for $n \geq 0$ and $|a_n| \leq r_1^{-n} M$ for $n < 0$ and the series converges uniformly on the compacta of $r_1 < |z| < R_1$. So, the series converges to a holomorphic function g on the annulus $r_1 < |z| < R_1$ by Corollary 54. Also f and g agree on $|z - z_0| < \frac{1}{2}s$ and therefore, they agree on $r_1 < |z| < R_1$. So (7.1) holds for $r_1 < |z| < R_1$.

Since we can choose r_1 as close to r (but with $r_1 > r$) as we please and R_1 as close to R (but with $R_1 < R$) as we please, this gives that (7.1) holds on $r < |z| < R$ and also that the series converges uniformly on the compacta of $r < |z| < R$.

We can also see that for all $n \in \mathbb{Z}$,

$$a_n = \frac{1}{2\pi i} \int \frac{f(z)}{z^{n+1}} dz$$

with the integral taken round $|z| = \rho$ anticlockwise is independent of ρ so long as $r < \rho < R$. This is because the circular paths corresponding to different ρ are homotopic and the integrand $z^{-n-1} f(z)$ is holomorphic in $r < |z| < R$. ■

The formula (7.2) shows that the coefficient of the Laurent expansion are related to the Fourier coefficient of the function restricted to a circle.

The uniqueness assertion is straightforward. Suppose that the series (7.1) converges uniformly on compacta to zero. Then we obtain by uniform convergence of integrals that for any ρ with $r < \rho < R$

$$0 = \lim_{N \to \infty} \int_0^{2\pi} \left(\sum_{n=-N}^{N} a_n \rho^n e^{in\theta} \right) \rho^{-k} e^{-ik\theta} d\theta = a_k$$

as required.

EXAMPLE Here is a simple example. Let

$$f(z) = \frac{1}{(z-1)(z+2)^2} = \frac{1}{9}\ (z-1)^{-1} - \frac{1}{3}\ (z+2)^{-2} - \frac{1}{9}\ (z+2)^{-1}$$

In the region $|z| < 1$ we write

$$f(z) = -\frac{1}{9}\ (1-z)^{-1} - \frac{1}{12}\ (1+z/2)^{-2} - \frac{1}{18}\ (1+z/2)^{-1}$$

$$= -\frac{1}{9}\sum_{n=0}^{\infty} z^n - \frac{1}{12}\sum_{n=0}^{\infty}(-1)^n(n+1)2^{-n}z^n - \frac{1}{18}\sum_{n=0}^{\infty}(-1)^n 2^{-n}z^n$$

giving

$$a_n = -\frac{1}{9} - \frac{1}{12}(-1)^n(n+1)2^{-n} - \frac{1}{18}(-1)^n 2^{-n}$$

for $n \geq 0$ and $a_n = 0$ for $n < 0$. For the region $1 < |z| < 2$, the series expansion for $(1-z)^{-1}$ is no good, we need to use

$$\begin{aligned} f(z) &= \frac{1}{9}\, z^{-1}\left(1 - z^{-1}\right)^{-1} - \frac{1}{12}\,(1 + z/2)^{-2} - \frac{1}{18}\,(1 + z/2)^{-1} \\ &= \frac{1}{9}\sum_{n=-\infty}^{-1} z^n - \frac{1}{12}\sum_{n=0}^{\infty}(-1)^n(n+1)2^{-n}z^n - \frac{1}{18}\sum_{n=0}^{\infty}(-1)^n 2^{-n}z^n \end{aligned}$$

giving

$$a_n = -\frac{1}{12}(-1)^n(n+1)2^{-n} - \frac{1}{18}(-1)^n 2^{-n}$$

for $n \geq 0$ and $a_n = \frac{1}{9}$ for $n < 0$. Finally, for $2 < |z|$, then we write

$$\begin{aligned} f(z) &= \frac{1}{9}\, z^{-1}\left(1 - z^{-1}\right)^{-1} - \frac{1}{3}\, z^{-2}\left(1 + 2z^{-1}\right)^{-2} - \frac{1}{9}\, z^{-1}\left(1 + 2z^{-1}\right)^{-1} \\ &= \frac{1}{9}\sum_{n=1}^{\infty} z^{-n} - \frac{1}{3}\, z^{-2}\sum_{n=0}^{\infty}(-1)^n(n+1)2^n z^{-n} - \frac{1}{9}\, z^{-1}\sum_{n=0}^{\infty}(-1)^n 2^n z^{-n} \end{aligned}$$

giving $a_n = 0$ for $n \geq 0$ and

$$a_n = \frac{1}{9} + \frac{1}{12}(-1)^n(n+1)2^{-n} + \frac{1}{18}(-1)^n 2^{-n}$$

for $n < 0$. This last expansion is a little bit misleading since the first two terms vanish. Numerically, it looks like

$$z^{-3} - 3z^{-4} + 9z^{-5} - 23z^{-6} + 57z^{-7} - 135z^{-8} + 313\, z^{-9} + \cdots$$

$\square$

EXAMPLE Find the Laurent expansion of $\operatorname{cosec}(z)$ in $0 < |z| < \pi$. since sin has a simple zero at $z = 0$, cosec has a simple pole. We can use the identity

$$\operatorname{cosec}(z) = \cot\left(\frac{z}{2}\right) - \cot(z)$$

and the expansion for cot (from one of the assignments) to get

$$\operatorname{cosec}(z) = z^{-1} + \sum_{k=1}^{\infty} (-1)^k \frac{(4^k - 2)B_{2k}}{(2k)!} z^{2k-1}$$

□

7.2 Residues and the Residue Theorem

First, let's state a more powerful version of the Cauchy Integral Theorem. We could have used this theorem in the proof of the existence of Laurent expansions.

THEOREM 68 *Let Ω be a bounded connected open subset of $\mathbb{C}$ with piecewise smooth boundary. Let $\partial\Omega$ denote the oriented boundary of Ω. Let $\zeta \in \Omega$ and f continuous on* $\mathrm{cl}(\Omega)$ *holomorphic in Ω. Then*

$$\int_{\partial\Omega} \frac{f(z)dz}{z - \zeta} = 2\pi i f(\zeta)$$

Proof. The idea is to cut a small closed disk D out of Ω. Let $U = \Omega \setminus D$. Then $\partial U = \partial\Omega - \partial D$. We apply the Green's Theorem version of Cauchy's Theorem to get

$$\int_{\partial U} \frac{f(z)dz}{z - \zeta} = 0$$

since $z \mapsto \dfrac{f(z)}{z - \zeta}$ is holomorphic in U. This gives

$$\int_{\partial\Omega} \frac{f(z)dz}{z - \zeta} = \int_{\partial D} \frac{f(z)dz}{z - \zeta} = 2\pi i f(\zeta)$$

from Theorem 36. ■

DEFINITION *Let ζ be an isolated singularity of a function f holomorphic in a punctured neighbourhood of ζ. Then f has a Laurent expansion about ζ. The coefficient of $(z-\zeta)^{-1}$ in this Laurent expansion is called the* ***residue*** *of f at ζ and will be denoted* $\mathrm{Res}(f,\zeta)$.

EXAMPLE Let $f(z)=\dfrac{z}{e^z-1-z}$. Then f has a simple pole at 0. This makes it easy to calculate the residue as $\lim_{z\to 0} zf(z)=2$. □

EXAMPLE The case of poles of higher order is trickier. For example let $g(z)=\dfrac{z}{e^z-1-z-\frac{1}{2}z^2}$. Then we will expand the denominator far enough

$$\begin{aligned} g(z) &= \frac{z}{\frac{1}{6}z^3+\frac{1}{24}z^4+\cdots} \\ &= 6z^{-2}(1+\frac{1}{4}z+\cdots)^{-1} \\ &= 6z^{-2}(1-\frac{1}{4}z+\cdots) \\ &= 6z^{-2}-\frac{3}{2}z^{-1}+\cdots \end{aligned}$$

and the residue is seen to be $-\frac{3}{2}$. □

EXAMPLE Even in case of an essential singularity, the concept of residue is still valid. For example, if $h(z)=\exp(z+z^{-1})$ we can calculate the residue as an integral

$$\mathrm{Res}(h,0)=\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\exp(e^{i\theta}+e^{-i\theta})d\theta=\sum_{k=0}^{\infty}\frac{1}{k!\,(k+1)!}=I_1(2)$$

where I_1 is one of the Bessel function family. □

EXAMPLE Find the residue of $\dfrac{e^{z^{-1}}}{1-z}$ at $z=0$. Clearly, $\dfrac{a}{1-z}$ is analytic in a neighbourhood of $z=0$. This means that the desired residue is also the residue of $\dfrac{e^{z^{-1}}-a}{1-z}$ at $z=0$ and we choose $a=e$ to kill the singularity at $z=1$. Then, with $w=z^{-1}$, we have

$$\frac{e^{z^{-1}}-e}{1-z}=w\frac{e^w-e}{w-1} \tag{7.3}$$

and $w \mapsto \dfrac{e^w - e}{w-1}$ is entire after we have resolved the removable singularity at $w = 1$. The value at $w = 0$ is $\dfrac{e^0 - e}{0-1} = e - 1$, this is also the coefficient of w in (7.3) and hence the desired residue. □

EXAMPLE Find the residue of $\dfrac{e^{z^{-1}}}{(1-z)^3}$ at $z = 0$. We need the coefficient of w in the Laurent expansion of $w^3 \dfrac{e^w}{(w-1)^3}$. So with $f(w) = e^w$, we have

$$f(1) + f'(1)(w-1) + \tfrac{1}{2}f''(1)(w-1)^2 = \tfrac{1}{2}e(w^2+1).$$

So, since the singularity of $\dfrac{f(w) - (f(1) + f'(1)(w-1) + \frac{1}{2}f''(1)(w-1)^2)}{(w-1)^3}$ is removable at $w = 1$ and the resulting function is entire, the coefficient of w in

$$w^3 \frac{f(w) - (f(1) + f'(1)(w-1) + \frac{1}{2}f''(1)(w-1)^2)}{(w-1)^3}$$

is zero. Hence the desired residue is also the residue of $\frac{1}{2}e\dfrac{z^2+1}{z^2(1-z)^3}$. But

$$\frac{z^2+1}{z^2(1-z)^3} = z^{-2} + 3z^{-1} - 2(z-1)^{-3} + 2(z-1)^{-2} - 3(z-1)^{-1}$$

and the desired residue is $\dfrac{3e}{2}$. □

Before tackling the Residue Theorem, we need the version that applies to a single singularity and a small circle centred at the singularity.

PROPOSITION 69 *Let f be continuous on the punctured disk $0 < |z - \zeta| \le r$ and holomorphic in $0 < |z - \zeta| < r$. Then*

$$\int_\Gamma f(z)dz = 2\pi i \operatorname{Res}(f, \zeta),$$

where Γ denotes the circle $|z - \zeta| = r$ traversed anticlockwise.

Proof. If we had taken $\Gamma(s)$ the circle $|z-\zeta| = s$ traversed anticlockwise with $0 < s < r$, then

$$\int_{\Gamma(s)} f(z)dz = 2\pi i \operatorname{Res}(f, \zeta)$$

would follow from the Laurent Expansion Theorem. It's enough to show that

$$\int_{\Gamma(s)} f(z)dz \longrightarrow \int_{\Gamma} f(z)dz$$

as $s \to r-$ and this is any easy consequence of the fact that f is uniformly continuous on $\{z \in \mathbb{C}; \frac{1}{2}r \leq |z| \leq r\}$. ■

THEOREM 70 *Let Ω be a bounded connected open subset of $\mathbb{C}$ with piecewise smooth boundary. Let $\partial\Omega$ denote the oriented boundary of Ω. Let f be continuous on $\operatorname{cl}(\Omega)$ holomorphic in $\Omega \setminus F$, where F is a finite set of singularities. Then*

$$\int_{\partial\Omega} f(z)dz = 2\pi i \sum_{\zeta \in F} \operatorname{Res}(f, \zeta)$$

Since the set of singularities is finite, it follows that each singularity is isolated.

Proof. The proof is similar to that of Theorem 68. For each $\zeta \in F$ make a small closed disk D_ζ, centred at ζ and such that all the disks are disjoint and contained in Ω. Let U be the set obtained from excising these disks from Ω. Then f is holomorphic on U and this yields

$$\int_{\partial\Omega} f(z)dz = \sum_{\zeta \in F} \int_{\partial D_\zeta} f(z)dz = 2\pi i \sum_{\zeta \in F} \operatorname{Res}(f, \zeta)$$

from proposition 69. ■

To get to the more advanced versions of the Residue Theorem, we will have to look again at the winding number.

PROPOSITION 71 *Let Γ be a piecewise C^1 loop in $\mathbb{C}$. Then for ζ in the unbounded connected component of $\mathbb{C} \setminus \Gamma$, $\operatorname{wind}_\Gamma(\zeta) = 0$.*

Proof. Since

$$\mathrm{wind}_\Gamma(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-\zeta}$$

it is easy to see that $|\,\mathrm{wind}_\Gamma(\zeta)\,| \longrightarrow 0$ as $\zeta \to \infty$. But $\mathrm{wind}_\Gamma(\cdot)$ takes integer values and is constant on the connected components of $\mathbb{C}\setminus\Gamma$. Hence the result.■

We now come to the winding number version of the Cauchy Integral Formula.

THEOREM 72 *Let Ω be an open subset of $\mathbb{C}$. Let $\Gamma_1,\ldots,\Gamma_m$ be piecewise C^1 loops in Ω such that*

$$W(\zeta) \equiv \sum_{k=1}^{m} \mathrm{wind}_{\Gamma_k}(\zeta) = 0 \quad \forall\zeta\in\mathbb{C}\setminus\Omega.$$

Let f be holomorphic in Ω. Then for $z \in \Omega\setminus\bigcup_{k=1}^m \Gamma_k$

$$\sum_{k=1}^{m}\int_{\Gamma_k}\frac{f(w)}{w-z}dw = 2\pi i\left(\sum_{k=1}^{m}\mathrm{wind}_{\Gamma_k}(z)\right)f(z) \tag{7.4}$$

Proof. Let

$$\varphi(z,w) = \begin{cases} \dfrac{f(w)-f(z)}{w-z} & \text{if } z\neq w,\\[2ex] f'(z) & \text{if } z = w.\end{cases}$$

then, it is easy to see that φ is continuous on $\Omega\times\Omega$ (use power series expansions to establish continuity at points on the diagonal) and holomorphic in each variable separately. Let

$$U = \{\zeta\in\mathbb{C};\zeta\notin\bigcup_{k=1}^{m}\Gamma_k, W(\zeta)=0\}$$

an open subset of $\mathbb{C}$ containing the set $\{\zeta\in\mathbb{C};|\zeta|>R\}$ for R sufficiently large and which by hypothesis satisfies $\Omega\cup U=\mathbb{C}$. Let

$$g(z) = \begin{cases} \displaystyle\sum_{k=1}^{m}\int_{\Gamma_k}\varphi(z,w)dw & \text{if } z\in\Omega,\\[3ex] \displaystyle\sum_{k=1}^{m}\int_{\Gamma_k}\frac{f(w)}{w-z}dw & \text{if } z\in U.\end{cases}$$

We will need to check that this is well-defined. In case $\zeta \in \Omega \cap U$ we have

$$\begin{aligned}\sum_{k=1}^{m} \int_{\Gamma_k} \left(\frac{f(w)}{w-z} - \varphi(z,w) \right) dw &= \sum_{k=1}^{m} \int_{\Gamma_k} \frac{f(z)}{w-z} dw \\ &= 2\pi i f(z) \sum_{k=1}^{m} \mathrm{wind}_{\Gamma_k}(\zeta) = 0\end{aligned}$$

It is easy to see that g is everywhere holomorphic (i.e. entire) and also that $g(z) \longrightarrow 0$ as $z \to \infty$ since we can use the definition for $\zeta \in U$ for this purpose. Since g is a bounded entire function, Liouville's Theorem (Theorem 42) asserts that g is constant and therefore zero. But then, repeating the above argument assuming that $z \in \Omega \setminus \bigcup_{k=1}^{m} \Gamma_k$, we get (7.4) as required. ■

COROLLARY 73 (WINDING NUMBER VERSION OF CAUCHY'S THEOREM) *Let Ω be an open subset of $\mathbb{C}$. Let $\Gamma_1, \ldots, \Gamma_m$ be piecewise C^1 loops in Ω such that*

$$\sum_{k=1}^{m} \mathrm{wind}_{\Gamma_k}(\zeta) = 0 \quad \forall \zeta \in \mathbb{C} \setminus \Omega.$$

Let h be holomorphic in Ω. Then for $z \in \Omega \setminus \bigcup_{k=1}^{m} \Gamma_k$

$$\sum_{k=1}^{m} \int_{\Gamma_k} h(w) dw = 0 \tag{7.5}$$

Proof. Choose $z \in \Omega$. Apply Theorem 72 with $f(w) = (w - z)h(w)$. ■

Finally, we now come to

COROLLARY 74 (WINDING NUMBER VERSION OF THE RESIDUE THEOREM) *Let Ω be an open subset of $\mathbb{C}$. Let $\Gamma_1, \ldots, \Gamma_m$ be piecewise C^1 loops in Ω such that*

$$\sum_{k=1}^{m} \mathrm{wind}_{\Gamma_k}(\zeta) = 0 \quad \forall \zeta \in \mathbb{C} \setminus \Omega.$$

Let f be holomorphic in $\Omega \setminus F$, where F is a finite set of singularities not meeting $\bigcup_{k=1}^{m} \Gamma_k$. Then

$$\sum_{k=1}^{m} \int_{\Gamma_k} f(w) dw = 2\pi i \sum_{z \in F} W(z) \operatorname{Res}(f, z) \tag{7.6}$$

where $W(z)$ has its usual meaning $W(z) = \sum_{k=1}^{m} \operatorname{wind}_{\Gamma_k}(z)$ as the total number of times that the contours wind about z.

Proof. For each $z \in F$, we strip out a small disk D_z. These disks are chosen so small that they avoid the contours Γ_k, are contained in Ω and avoid each other. We would like to replace Ω by $\Omega_1 = \Omega \setminus \bigcup_{z \in F} D_z$. We have that f is holomorphic on Ω_1, but we have messed up the winding number condition. To fix this, for each $z \in F$ we introduce an additional loops L_z winding around each of the disk D_z $-W(z)$ times. The loop L_z is located in Ω_1 so very close to ∂D_z that it does not interfere with anything else. Applying Corollary 7.5 we now get

$$\sum_{k=1}^{m} \int_{\Gamma_k} f(w)dw + \sum_{z \in D} \int_{L_z} f(w)dw = 0$$

Proposition 69 now gives

$$\int_{L_z} f(w)dw = 2\pi i \operatorname{Res}(f, z) \operatorname{wind}_{L_z}(z) = -2\pi i \operatorname{Res}(f, z) W(z)$$

and (7.6) follows. ∎

7.3 Method of Residues for Evaluating Definite Integrals

This section is all examples.

EXAMPLE We wish to evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x} dx$. We proceed by integrating a function related to the given one around a contour. Ler $r > 0$ be very small and $R > 0$ be very large. In this case, we choose $f(z) = \dfrac{e^{iz}}{z}$ and integrate around the contour comprising four sections

1. Along the real axis from r to R.
2. Around the semicircle $\theta \mapsto Re^{i\theta}$ from $\theta = 0$ to $\theta = \pi$.
3. Along the real axis from $-R$ to $-r$.
4. Around the semicircle $\theta \mapsto re^{i\theta}$ from $\theta = \pi$ to $\theta = 0$.

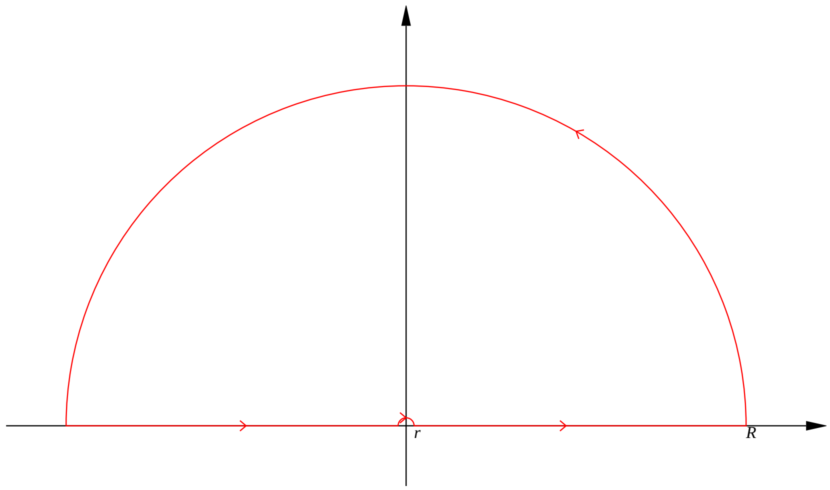

Figure 7.2: Contour for $\displaystyle\int_0^\infty \frac{\sin x}{x} dx$.

This is a C^1 loop and f has no singularities "inside" the loop. The only singularity of f is at $z = 0$ and the contour has winding number zero about $z = 0$. So, we obtain

$$\begin{aligned} 0 &= \int_r^R \frac{e^{ix}}{x} dx + \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta} d\theta + \int_{-R}^{-r} \frac{e^{ix}}{x} dx + \int_\pi^0 \frac{e^{ire^{i\theta}}}{re^{i\theta}} ire^{i\theta} d\theta \\ &= \int_r^R \frac{e^{ix}}{x} dx + i \int_0^\pi e^{iRe^{i\theta}} d\theta - \int_r^R \frac{e^{-ix}}{x} dx - i \int_0^\pi e^{ire^{i\theta}} d\theta \\ &= 2i \int_r^R \frac{\sin x}{x} dx + i \int_0^\pi e^{iR\cos(\theta)} e^{-R\sin(\theta)} d\theta - i \int_0^\pi e^{ire^{i\theta}} d\theta \end{aligned}$$

Obviously, we are going to let $r \to 0+$ and $R \to \infty$. As $r \to 0+$, we have $e^{ire^{i\theta}} \to 1$ uniformly in θ (since $re^{i\theta} \to 0$ uniformly in θ) and so

$$\int_0^\pi e^{ir\cos(\theta)} e^{-r\sin(\theta)} d\theta \underset{r\to 0+}{\longrightarrow} \pi$$

We also have

$$\int_0^\pi e^{iR\cos(\theta)} e^{-R\sin(\theta)} d\theta \underset{R\to\infty}{\longrightarrow} 0$$

but this is less obvious. To justify this we proceed by

$$\begin{aligned}
\left| \int_0^\pi e^{iR\cos(\theta)} e^{-R\sin(\theta)} d\theta \right| &\leq \int_0^\pi e^{-R\sin(\theta)} d\theta \\
&\leq \int_0^{R^{-\frac{1}{2}}} d\theta + \int_{R^{-\frac{1}{2}}}^{\pi - R^{-\frac{1}{2}}} e^{-R\sin(\theta)} d\theta + \int_{\pi - R^{-\frac{1}{2}}}^{\pi} d\theta \\
&\leq 2R^{-\frac{1}{2}} + \pi e^{-R\sin(R^{-\frac{1}{2}})} \underset{R\to\infty}{\longrightarrow} 0
\end{aligned}$$

Finally, this gives

$$2i \int_0^\infty \frac{\sin x}{x} dx = \pi i$$

and we conclude

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

□

The next example is computationally more intensive.

EXAMPLE Find $\displaystyle\int_0^{2\pi} \frac{d\theta}{32 - 18\cos\theta + 5\sin 2\theta}$. here we will put $z = e^{i\theta}$ and integrate over the unit circle anticlockwise — call it Γ. We get $dz = ie^{i\theta} d\theta$ and it follows that $d\theta = -iz^{-1} dz$. So, the integral becomes

$$\begin{aligned}
& -i \int_\Gamma \frac{dz}{z(32 - 9z - 9z^{-1} - \frac{5}{2}iz^2 + \frac{5}{2}iz^{-2})} \\
= & -i \int_\Gamma \frac{zdz}{-\frac{5}{2}iz^4 - 9z^3 + 32z^2 - 9z + \frac{5}{2}i}
\end{aligned}$$

The roots of the denominator are at $-\frac{6}{5} + \frac{12}{5}i + \frac{1}{5}\sqrt{31} - \frac{2}{5}i\sqrt{31}$, $-\frac{6}{5} + \frac{12}{5}i - \frac{1}{5}\sqrt{31} + \frac{2}{5}i\sqrt{31}$, $\frac{2}{5} - \frac{1}{5}i$, and $2 - i$. The first and third roots listed are in the unit circle and the second and fourth are outside it. Only the residues coming from the first and third roots will contribute. So the integral is

$$(2\pi i)(-i)\left(\frac{\alpha}{g'(\alpha)} + \frac{\beta}{g'(\beta)}\right) = \frac{\pi}{3270}(217 + 13\sqrt{31})$$

where α and β stand for the first and third roots and $g(z) = -\frac{5}{2}iz^4 - 9z^3 + 32z^2 - 9z + \frac{5}{2}i$. □

The next example is useful in the theory of Fourier integrals and can be used to establish the Plancherel Theorem and Inversion formula for Fourier Integrals.

EXAMPLE We start with the formula

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}$$

which is proved in the MATH 255 notes. Unfortunately, there does not seem to be any way to establish this with contour integrals. What we would like to evaluate is

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-iux}dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+iu)^2} e^{-i\frac{1}{2}u^2}dx$$

for u real. We will show that $\displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+iu)^2}dx = \sqrt{2\pi}$ and it follows that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} e^{-iux}dx = \sqrt{2\pi}e^{-i\frac{1}{2}u^2}$$

So here, we take $f(z) = e^{-\frac{1}{2}z^2}$ and integrate around the rectangle from $-R$ to

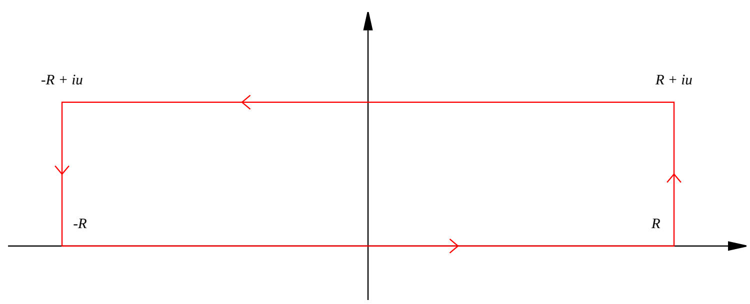

Figure 7.3: Rectangular contour for Fourier Integral example.

R along the real axis, from R to $R + iu$ along a vertical line, then from $R + iu$ to $-R + iu$ horizontally and finally from $-R + iu$ to $-R$ vertically. Cauchy's Theorem yields

$$\int_{-R}^{R} e^{-\frac{1}{2}x^2}dx + \int_{0}^{u} e^{-\frac{1}{2}(R+it)^2}idt - \int_{-R}^{R} e^{-\frac{1}{2}(x+iu)^2}dx - \int_{0}^{u} e^{-\frac{1}{2}(-R+it)^2}idt = 0$$

We estimate the second and fourth integrals by

$$\left|\int_0^u e^{-\frac{1}{2}(\pm R+it)^2}dt\right| = \left|\int_0^u e^{-\frac{1}{2}R^2}e^{\mp iRt}e^{\frac{1}{2}t^2}dt\right| \le |u|e^{\frac{1}{2}u^2}e^{-\frac{1}{2}R^2}$$

Both these integrals tend to zero as $R \to \infty$ while keeping u constant. Thus, letting $R \to \infty$, we get

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+iu)^2}dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}dx$$

as required.

The application of this (with most of the details beyond the scope of this course) is to the Fourier integral of a function f defined on the line, continuous and with compact support say.

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{-iux}f(x)dx$$

is the Fourier transform of f defined for u real. Then for $t > 0$

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(u)|^2 e^{-\frac{1}{2}t^2u^2}du &= \frac{1}{2\pi}\iiint e^{-iux}e^{iuy}e^{-\frac{1}{2}t^2u^2}f(x)\overline{f(y)}dxdydu \\
&= \frac{1}{2\pi}\iiint e^{-iux}e^{iuy}e^{-\frac{1}{2}t^2u^2}duf(x)\overline{f(y)}dxdy \\
&= \frac{1}{\sqrt{2\pi}}\iint t^{-1}e^{-\frac{1}{2}t^{-2}(x-y)^2}f(x)\overline{f(y)}dxdy
\end{aligned}$$

Letting $t \to 0+$ we get Plancherel's Theorem

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(u)|^2du = \int_{-\infty}^{\infty}|f(x)|^2dx$$

using among other things, the fact that $k_t(x,y) = \dfrac{1}{\sqrt{2\pi}}t^{-1}e^{-\frac{1}{2}t^{-2}(x-y)^2}$ is a summability kernel on the line as $t \to 0+$. $\square$

EXAMPLE Consider $\displaystyle\int_0^\infty \frac{x^a}{(1+x)^2}dx$ for $-1 < a < 1$. The integral has to be treated as an improper integral at the upper limit and in case $-1 < a < 0$ also as an improper integral at the lower limit. The integrand looks like x^a for small x, so the condition $a > -1$ is needed for the integral to converge near $x = 0$. The integrand looks like x^{a-2} for large x, so the condition $a < 1$ is needed for convergence at infinity.

Now the standard way of approaching this is to integrate the function

$$f(z) = \frac{z^a}{(1+z)^2}$$

around the famous keyhole contour. Doing this depends on selecting a non-standard branch of the power z^a, namely one that has a cut along the positive real axis. The conceptual difficulties that arise can be avoided by making a substitution in the original integral before considering how to embed the problem in the complex domain. Our first step is to put $x = e^t$ and then the desired integral becomes

$$\int_{-\infty}^\infty \frac{e^{at}}{e^t + 2 + e^{-t}}dt.$$

We now take $f(z) = \dfrac{e^{az}}{e^z + 2 + e^{-z}}$ and we integrate over the rectangular contour from $-R$ to R along the real axis, from R to $R+2\pi i$ along a vertical line, then from $R+2\pi i$ to $-R+2\pi i$ horizontally and finally from $-R+2\pi i$ to $-R$ vertically. In some obvious sense, this is equivalent to integrating the original function around the keyhole contour.

The function f has its poles, where $(1+\cosh(z)) = 2\cosh^2\left(\frac{z}{2}\right)$ has its zeros and the only zero of $1+\cosh(z)$ inside our new contour is at $i\pi$. To find the residue, substitute $z = i\pi + w$. We find

$$\begin{aligned}
f(z) &= e^{i(1+a)\pi}e^{(1+a)w}(e^w - 1)^{-2} \\
&= e^{i(1+a)\pi}e^{(1+a)w}(w + \tfrac{1}{2}w^2 + \cdots)^{-2} \\
&= e^{i(1+a)\pi}w^{-2}(1 + (1+a)w + \cdots)(1 + \tfrac{1}{2}w + \cdots)^{-2} \\
&= e^{i(1+a)\pi}w^{-2}(1 + aw + \cdots)
\end{aligned}$$

and the residue is seen to be $ae^{i(1+a)\pi}$.

Thus, we get

$$2\pi i a e^{i(1+a)\pi} = \int_{-R}^R \frac{e^{at}}{e^t + 2 + e^{-t}}dt + \int_0^{2\pi} \frac{e^{a(R+is)}}{e^{R+is} + 2 + e^{-R-is}}ids$$

$$- \int_{-R}^{R} \frac{e^{a(t+2i\pi)}}{e^t + 2 + e^{-t}} dt - \int_0^{2\pi} \frac{e^{a(-R+is)}}{e^{-R+is} + 2 + e^{R-is}} ids$$

As $R \to \infty$, the second and fourth integrals tend to zero giving

$$2\pi i a e^{i(1+a)\pi} = \int_{-\infty}^{\infty} \frac{e^{at}}{e^t + 2 + e^{-t}} dt - \int_{-\infty}^{\infty} \frac{e^{a(t+2i\pi)}}{e^t + 2 + e^{-t}} dt$$

which simplifies to

$$\int_{-\infty}^{\infty} \frac{e^{at}}{e^t + 2 + e^{-t}} dt = \frac{2\pi i a e^{i(1+a)\pi}}{1 - e^{2ia\pi}} = \frac{2\pi i a}{e^{ia\pi} - e^{-ia\pi}} = \frac{\pi a}{\sin(\pi a)}$$

□

EXAMPLE Consider $\displaystyle\int_0^{\infty} \frac{x^a}{1+x^2} dx$ for $-1 < a < 1$. The integral has to be treated as an improper integral at the upper limit and in case $-1 < a < 0$ also as an improper integral at the lower limit. The integrand looks like x^a for small x, so the condition $a > -1$ is needed for the integral to converge near $x = 0$. The integrand looks like x^{a-2} for large x, so the condition $a < 1$ is needed for convergence at infinity.

Our first step is to put $x = e^t$ and then the desired integral becomes

$$\int_{-\infty}^{\infty} \frac{e^{at}}{e^t + e^{-t}} dt.$$

We now take $f(z) = \dfrac{e^{az}}{e^z + e^{-z}}$ and we integrate over the rectangular contour from $-R$ to R along the real axis, from R to $R + \pi i$ along a vertical line, then from $R + \pi i$ to $-R + \pi i$ horizontally and finally from $-R + \pi i$ to $-R$ vertically.

The function f has its poles, where cosh has its zeros and the only zero of cosh inside our new contour is at $i\pi/2$. The residue is $\dfrac{e^{ia\pi/2}}{e^{i\pi/2} - e^{-i\pi/2}} = -\frac{1}{2} i e^{ia\pi/2}$. Thus, we get

$$\pi e^{ia\pi/2} = \int_{-R}^{R} \frac{e^{at}}{e^t + e^{-t}} dt + \int_0^{\pi} \frac{e^{a(R+is)}}{e^{R+is} + e^{-R-is}} ids$$

$$- \int_{-R}^{R} \frac{e^{a(t+i\pi)}}{-e^t - e^{-t}} dt - \int_0^{\pi} \frac{e^{a(-R+is)}}{e^{-R+is} + e^{R-is}} ids$$

As $R \to \infty$, the second and fourth integrals tend to zero giving

$$\pi e^{ia\pi/2} = 2\pi i\Big(- \tfrac{1}{2} i e^{ia\pi/2}\Big) = \int_{-\infty}^{\infty} \frac{e^{at}}{e^t + e^{-t}} dt + \int_{-\infty}^{\infty} \frac{e^{a(t+i\pi)}}{e^t + e^{-t}} dt$$

which simplifies to

$$\int_{-\infty}^{\infty} \frac{e^{at}}{e^t + e^{-t}} dt = \frac{\pi e^{ia\pi/2}}{1 + e^{ia\pi}} = \frac{\pi}{2} \sec\left(\frac{a\pi}{2}\right)$$

□

7.4 Singularities in Several Complex Variables

The situation as regards isolated singularities in several variables is bizarre. First of all, we need to agree on what a holomorphic function is. We give two definitions, one minimal and the other maximal and in fact, they agree. We will leave this as an exercise.

DEFINITION *Let $\Omega \subseteq \mathbb{C}^d$. Let $f : \Omega \longrightarrow \mathbb{C}$. Then*

- *f is holomorphic if it is continuous and holomorphic in each variable separately.*
- *f is analytic if for each $\zeta \in \Omega$, there is a neighbourhood U of ζ in Ω and $a_n \in \mathbb{C}$ for $n \in \mathbb{Z}^{+d}$ such that the series*

$$\sum_{n \in \mathbb{Z}^{+d}} a_n (z - \zeta)^n$$

converges unconditionally uniformly on the compacta of U to $f(z)$.

In the second definition, n is a multiindex and we interpret

$$(z - \zeta)^n = \prod_{j=1}^{d} (z_j - \zeta_j)^{n_j}.$$

The result that we wish to prove here is the following.

PROPOSITION 75 *Let* $\Omega = \{(z_1, z_2) \in \mathbb{C}^2; 1 < |z_1|^2 + |z_2|^2 < 9\}$. *Let* f *be holomorphic in* Ω. *Then* f *extends to a holomorphic function in* $\{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 < 9\}$.

In other words, the whole of the inner ball is a removable singularity!

Proof.
For $|z_1|, |z_2| < 2$, let us define

$$g(z_1, z_2) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(2e^{i\theta}, z_2)}{2e^{i\theta} - z_1} 2e^{i\theta} d\theta$$

It is easy to see that g is continuous and holomorphic in each variable separately (use differentiations under the integral sign) on $\{(z_1, z_2) \in \mathbb{C}^2; |z_1| < 2, |z_2| < 2\}$. Now suppose that $1 < |z_2| < 2$, so that $z_1 \mapsto f(z_1, z_2)$ is holomorphic on $\{z_1; |z_1| < \sqrt{9 - |z_2|^2}\}$. Then the Cauchy Integral Formula shows that $f(z_1, z_2) = g(z_1, z_2)$. Now free z_2 and fix z_1 such that $|z_1| < 2$. Then $f(z_1, z_2) = g(z_1, z_2)$ for $\sqrt{1 - |z_1|^2} < |z_2| < 2$ in case $|z_1| \le 1$ and for $|z_2| < 2$ in case $|z_1| > 1$ since it is already known to hold for $1 < |z_2| < 2$. We have just shown that f and g agree where both are defined. It follows that we can fabricate a glued function h holomorphic on the union of the domains of definition. But this is just $\{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 < 9\}$ as required. ■

In the Proposition above, we have chosen the domain between balls or radius 1 and 3, but the same result is true for the domain between balls of radius r and R so long as $r < R$.

8

Variation of the Argument and Rouché's Theorem

We will start with the path lifting lemma

LEMMA 76 *Let* $f : [0,1] \longrightarrow \mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ *be continuous. Then* f *possesses a continuous lift, i.e. a continuous mapping* $\tilde{f} : [0,1] \longrightarrow \mathbb{R}$ *such that* $f(t) = e^{i\tilde{f}(t)}$ *for all* $t \in [0,1]$. *Furthermore, if* $\tilde{f}(0)$ *is specified (satisfying* $e^{i\tilde{f}(0)} = f(0)$*), then* $\tilde{f}$ *is uniquely determined.*

Proof. It is clear that if f maps into a closed interval on $\mathbb{T}$, then the result is obvious. This is because for every such interval K, there is an interval I in the line such that the map $s \mapsto e^{is}$ is a bijection from I onto K continuous in both directions. We now handle the general case. Since f is continuous on $[0,1]$ it is uniformly continuous. So there exists $\delta > 0$ such that $\omega_f(\delta) < \sqrt{2}$. So every closed interval of $[0,1]$ of length less than δ gets mapped onto an interval in $\mathbb{T}$. We can therefore break up

$$[0,1] = \bigcup_{k=1}^{n} \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

where $n\delta < 1$ and then for each k there will exist a continuous lift g_k defined on $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ such that $e^{ig_k} = f$ on this interval. Now

$$e^{ig_k(k/n)} = f(k/n) = e^{ig_{k+1}(k/n)}$$

and it follows that $g_{k+1}(k/n) = g_k(k/n) + 2m_k\pi$, for some integer m_k. We then adjust the lifts by setting new continuous lifts

$$h_k(t) = g_k(t) - 2\pi \left(\sum_{j=1}^{k-1} m_j\right) \qquad t \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

which are coherent in the sense $h_{k+1}(k/n) = h_k(k/n)$. The gluing lemma allows the h_k to be glued into a single continuous function $\tilde{f}$ which satisfies the required conditions.

The uniqueness assertion amounts to showing that if $g : [0,1] \longrightarrow \mathbb{R}$ is continuous $e^{ig} = 1$ and $g(0) = 0$, then g is identically zero. Since g takes values in $2\pi\mathbb{Z}$ and g is continuous, it must be constant. ■

This is of course the start of some interesting topology. We say that a path connected metric space X is ***simply connected*** if every continuous loop in X is homotopic to a constant map. We will leave the following theorem as an exercise.

THEOREM 77 *Let X be a path connected, locally path connected, simply connected metric space. Let $f : X \longrightarrow \mathbb{T}$ be continuous. Then f possesses a continuous lift $\tilde{f} : X \longrightarrow \mathbb{R}$.*

Before we can prove this result, we will need several lemmas.

LEMMA 78 *Let X be a simply connected metric space. Let α, β be two paths in X (i.e. continuous maps from $[0,1]$ to X) such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Then α and β are homotopic via a homotopy that respects the endpoints. Explicitly, there is a continuous map $H : [0,1] \times [0,1] \longrightarrow X$ such that $H(0,t) = \alpha(t)$, $H(1,t) = \beta(t)$, $H(t,0) = \alpha(0)$ and $H(t,1) = \alpha(1)$ for all $t \in [0,1]$.*

Proof. We make a loop γ by adjoining α with the reversal of β, explicitly

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \beta(2-2t) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Note that $\gamma(\frac{1}{2})$ is well-defined and that $\gamma(0) = \gamma(1)$. Let K be the homotopy linking γ to a constant loop with value ξ. We have $K(0,t) = \gamma(t)$ and $K(1,t) = \xi$ for all $t \in [0,1]$ and also we have $K(s,0) = K(s,1)$ for all $s \in [0,1]$ since this is a homotopy of loops. We now cut this mapping apart and reassemble it.

$$L(s,t) = \begin{cases} K(2s, \frac{1}{2}t) & \text{if } 0 \le s \le \frac{1}{2}, \\ K(2-2s, 1-\frac{1}{2}t) & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

The definitions agree for $s = \frac{1}{2}$ since $K(1,t) = \xi$ for all $t \in [0,1]$. Also for $0 \le s \le \frac{1}{2}$,

$$L(s,0) = K(2s,0) = K(2s,1) = L(1-s,0),$$
$$L(s,1) = K(2s, \tfrac{1}{2}) = L(1-s,1).$$

Unfortunately L does not satisfy the necessary requirements, because for $s \mapsto L(s,0)$ and $s \mapsto L(s,1)$ are not constant. We fix this by waisting L. That is, we define

$$H(s,t) = \begin{cases} L(r,0) & \text{for } 0 \le t \le s(1-s) \text{ and where } r(1-r) = t, \\ L(r,1) & \text{for } 0 \le 1-t \le s(1-s) \text{ and where } r(1-r) = 1-t, \\ L(s,r) & \text{for } s(1-s) \le t \le 1-s(1-s) \\ & \text{and where } (1-r)s(1-s) + r(1-s(1-s)) = t. \end{cases}$$

One can check that H is continuous by using the glueing lemma. We note also that the choice of root r satisfying $r(1-r) = t$ in the first case is irrelevant because if r is a root, then the other root is $1-r$ and we know $L(r,0) = L(1-r,0)$. Similarly for the second case. We have $H(s,0) = L(0,0) = K(0,0) = \alpha(0)$ for all $s \in [0,1]$ and similarly $H(s,1) = \alpha(1)$ for all $s \in [0,1]$. On the other hand $H(0,t) = L(0,t) = K(0, \frac{1}{2}t) = \alpha(t)$ and $H(1,t) = L(1,t) = K(0, 1-\frac{1}{2}t) = \beta(t)$ for all $t \in [0,1]$. ■

LEMMA 79 *Let α and β be homotopic paths in $\mathbb{T}$ (via a homotopy that respects the endpoints). We are assuming that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts to $\mathbb{R}$ such that $\tilde{\alpha}(0) = \tilde{\beta}(0)$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$.*

Sketch proof. If α and β are uniformly close, say $|\alpha(t) - \beta(t)| < \sqrt{2}$ for all $t \in [0,1]$, then $|1 - \alpha(t)^{-1}\beta(t)| < \sqrt{2}$ or $\Re(\alpha(t)^{-1}\beta(t)) > 0$. It is then easy to lift this path to a path in $]-\frac{1}{2}\pi, \frac{1}{2}\pi[$ in $\mathbb{R}$. The lifted path therefore takes the value zero at $t = 1$. The uniqueness assertion of Lemma 76 shows that $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

In general, α and β are not uniformly close. Let $H : [0,1] \times [0,1] \longrightarrow \mathbb{T}$ be the homotopy between them satisfying $H(0,t) = \alpha(t)$, $H(1,t) = \beta(t)$, $H(t,0) = \alpha(0)$ and $H(t,1) = \alpha(1)$ for all $t \in [0,1]$. Then H is uniformly continuous and for $k = 1, 2, \ldots, n$, the paths $t \mapsto H(\frac{k-1}{n}, t)$ and $t \mapsto H(\frac{k}{n}, t)$ are uniformly close. The result follows. ■

Sketch proof of Theorem 77. Let $f : X \longrightarrow \mathbb{T}$. Fix a point $x_0 \in X$. Let $\omega_0 = f(x_0) \in \mathbb{T}$ and find $t_0 \in \mathbb{R}$ such that $\omega_0 = e^{it_0}$. Now for every $x_1 \in X$, find a continuous path α from x_0 to x_1, possible since X is path connected. Then $f \circ \alpha$ is a path in $\mathbb{T}$ which therefore has a unique lift $\widetilde{f \circ \alpha}$ such that $\widetilde{f \circ \alpha}(0) = t_0$. Then we define $\tilde{f}(x_1) = \widetilde{f \circ \alpha}(1)$.

Now if β is another continuous path from x_0 to x_1, then by Lemma 78, α and β are homotopic via a homotopy that respects the endpoints. The same is then true for $f \circ \alpha$ and $f \circ \beta$. It then follows from Lemma 79 that $\widetilde{f \circ \alpha}(1) = \widetilde{f \circ \beta}(1)$. This shows that $\tilde{f}(x_1)$ is independent of the path chosen.

It remains to show that $\tilde{f}$ is continuous. Let $\pi > \epsilon > 0$, then the open neighbourhood $]\tilde{f}(x_1) - \epsilon, \tilde{f}(x_1) + \epsilon[$ of $\tilde{f}(x_1)$ lives above a interval neighbourhood V of $f(x_1)$ in $\mathbb{T}$. Since X is locally path connected and f is continuous, we can find a path connected neighbourhood U of x_1 in X such that $f(U) \subseteq V$. Let $x_2 \in U$. Let γ be a continuous path from x_1 to x_2 lying entirely in U. Then we make a continuous path from x_0 to x_2 by adjoining α and γ to produce a new path $\eta = \alpha \cdot \gamma$. Then $f \circ \eta = (f \circ \alpha) \cdot (f \circ \gamma)$. Then $\widetilde{f \circ \eta} = (\widetilde{f \circ \alpha}) \cdot (\widetilde{f \circ \gamma})$ and we can compute $\widetilde{f \circ \gamma}$ directly by lifting from V to $]\tilde{f}(x_1) - \epsilon, \tilde{f}(x_1) + \epsilon[$. Thus

$$\tilde{f}(x_2) = \widetilde{f \circ \eta}(1) = \widetilde{f \circ \gamma}(1) \in]\tilde{f}(x_1) - \epsilon, \tilde{f}(x_1) + \epsilon[$$

This shows that $\tilde{f}$ is continuous. ■

8.1 Meromorphic Functions

Here is another definition.

DEFINITION *Let Ω be a connected open subset of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}_\infty$. Then f is **meromorphic** if it is not identically infinite and is holomorphic as a map from Ω to $\mathbb{C}_\infty$ in the sense of complex manifolds. Thus f is allowed to have isolated singularities (they cannot accumulate in Ω since then $1/f$ would have a non-isolated zero and f would be identically infinite). Each isolated singularity is necessarily a pole corresponding to the zeros of $1/f$. Both the zeros and poles of f can accumulate on the boundary of Ω.*

8.2 Variation of the Argument

THEOREM 80 (VARIATION OF THE ARGUMENT) *Let Ω be a connected open subset of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}_\infty$ be meromorphic. Let Γ be a closed piecewise C^1 loop in Ω not passing through any zero or pole of f. Then*

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)} dz = \sum_\zeta \mathrm{wind}_\Gamma(\zeta)\, m(\zeta)$$

where the sum on the right is over the zeros and poles of f. If ζ is a zero, then $m(\zeta)$ is the order of ζ as a zero of f. If ζ is a pole, then $-m(\zeta)$ is the order of ζ as *a pole of f. Since* $\mathrm{wind}_\Gamma(\zeta) = 0$ *for all* ζ *in the unbounded component of* $\mathbb{C} \setminus \Gamma$, *it follows that all but finitely many terms in the sum vanish.*

Proof. The function $z \mapsto \dfrac{f'(z)}{f(z)}$ is holomorphic except at the zeros and poles of f. If ζ is such a zero or pole, we write $f(z) = (z-\zeta)^m g(z)$ where $g(\zeta) \neq 0$ and where g is holomorphic in a neighbourhood of ζ. The we get locally near ζ

$$\frac{f'(z)}{f(z)} = \frac{m}{z-\zeta} + \frac{g'(z)}{g(z)}$$

and $z \mapsto \dfrac{g'(z)}{g(z)}$ is holomorphic in this neighbourhood. The result now follows from the Residue Theorem. ■

This result is called the variation of the argument because of the way that $\dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f'(z)}{f(z)} dz$ can be interpreted. Let $f(\Gamma)$ be the C^1 loop in $f(\Omega)$ defined by $t \mapsto f(z(t))$ where Γ is the loop defined by $t \mapsto z(t)$. Then a change of variables gives

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\int_{f(\Gamma)} \frac{1}{w} dw = \mathrm{wind}_{f(\Gamma)}(0)\,.$$

COROLLARY 81 *Under the same hypotheses as Theorem 80, we have*

$$\sum_\zeta \mathrm{wind}_\Gamma(\zeta)\, m(\zeta) = \mathrm{wind}_{f(\Gamma)}(0),$$

where the sum on the left is over the zeros and poles of f and is interpreted in the same way.

8.3 Rouché's Theorem

It is now possible to establish Rouché's Theorem, which is a seat of the pants way of counting zeros.

THEOREM 82 *Let Ω be a connected open subset of $\mathbb{C}$ and $f, g : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Let Γ be a closed piecewise C^1 loop in Ω not passing through any zero of f. Further, suppose that $|g(z)| < |f(z)|$ for all $z \in \Gamma$. Then* $\mathrm{wind}_{f(\Gamma)}(0) = \mathrm{wind}_{h(\Gamma)}(0)$ *where $h = f + g$ and consequently*

$$\sum_{\zeta \in f^{-1}(\{0\})} \mathrm{wind}_\Gamma(\zeta)\, m_f(\zeta) = \sum_{\zeta \in h^{-1}(\{0\})} \mathrm{wind}_\Gamma(\zeta)\, m_h(\zeta).$$

Proof. We have

$$h(z) = f(z) + g(z) = f(z)\left(1 + \frac{g(z)}{f(z)}\right)$$

$$\Re\left(\frac{h(z)}{f(z)}\right) = \Re\left(1 + \frac{g(z)}{f(z)}\right) > 0$$

where we have used the fact $|w| < 1 \Longrightarrow \Re(1 + w) > 0$. Now parametrize the loop Γ by $t \mapsto z(t)$. Then let

$$\mathrm{sgn}(f(z(t)) = e^{i\theta(t)} \qquad \text{and} \qquad \mathrm{sgn}(h(z(t)) = e^{i\varphi(t)}$$

and we find

$$\Re(e^{i\varphi(t)}e^{-i\theta(t)}) > 0$$

and it follows from

$$\{s \in \mathbb{R}; \Re e^{is} > 0\} = \bigcup_{m \in \mathbb{Z}}](2m - \tfrac{1}{2})\pi, (2m + \tfrac{1}{2})\pi[,$$

the fact that $\varphi - \theta$ is continuous and the Intermediate Value Theorem that there is an integer m such that $(2m - \frac{1}{2})\pi < \varphi(t) - \theta(t) < (2m + \frac{1}{2})\pi$ for all t. It follows that $-2\pi < (\varphi(1) - \theta(1)) - (\varphi(0) - \theta(0)) < 2\pi$. But the quantity in the middle is an integer multiple of 2π since $f(z(0)) = f(z(1))$ and $h(z(0)) = h(z(1))$ and so it follows that $\mathrm{wind}_{f(\Gamma)}(0) = \mathrm{wind}_{h(\Gamma)}(0)$ as required. ■

EXAMPLE Let $h(z) = 10 + 7z^2 + 2z^3$. Then on $|z| = 1$ we take $f = 10$, $g = 7z^2 + 2z^3$, we get $|f| = 10$, $|g| \le 9 < 10$. So h has no zeros in $|z| < 1$. Then, on $|z| = 2$ we take $f = 7z^2$ and $g = 10 + 2z^3$, we get $|f| = 28$ and $|g| \le 26 < 28$. So h has exactly two zeros in $|z| < 2$. Then, on $|z| = 4$ we take $f = 2z^3$ and $g = 10 + 7z^2$, we get $|f| = 128$ and $|g| \le 122 < 128$. So h has exactly three zeros in $|z| < 4$.

Using other facts one can say considerably more. The single root in $2 < |z| < 4$ must be real since otherwise there would be at least two roots with the same radius. Clearly this root is negative and you can locate it easily with a search as lying between -3.84 and -3.83. The product of all roots is positive, so if the remaining roots are real, then one is positive and one is negative. This is impossible, since h is increasing on the positive axis and $h(0) = 10$. So the it is a complex conjugate pair of roots $a + ib$ that lie in $1 < |z| < 2$. Since the sum and product of all three roots are known, it is easy to find approximate locations. □

EXAMPLE Consider the power series

$$f(z) = \sum_{n=0}^{\infty} 3^{-n^2} z^n$$

which clearly has infinite radius and defines an entire function. Consider the circle $|z| = 9^k$. On this circle, $|3^{-k^2} z^k| = 3^{k^2}$. On the other hand

$$\left| \sum_{n \neq k} 3^{-n^2} z^n \right| \leq \sum_{n \neq k} 3^{-(n-k)^2} 3^{k^2} \leq 3^{k^2} \cdot 2 \sum_{m=1}^{\infty} 3^{-m^2} < 3^{k^2}$$

It follows that for $k = 0, 1, 2, \ldots$ f has exactly k zeros in $|z| < 9^k$. Further analysis shows that all the zeros are real and negative. □

EXAMPLE Suppose that a and b are real constants such that $|b| < 1$ and $|a \pm b\frac{\pi}{2}| < 1$. We will show that $h(z) = \sin(z) - (a + bz)$ has only one zero in $-\frac{\pi}{2} < \Re z < \frac{\pi}{2}$, namely the one on the real axis. The idea is to apply Rouché's Theorem on a rectangular contour with sides corresponding to $x = \pm\frac{\pi}{2}$ and $y = \pm Y$ where Y is a large number. We take $f(z) = \sin(z)$ and $g(z) = a + bz$. We have $|f(x + iy)|^2 = (\sin x)^2 + (\sinh y)^2$ and $|g(x + iy)|^2 = (a + bx)^2 + b^2y^2$ for x and y real. On a side of the form $z = \pm\frac{\pi}{2} + iy$ with $-Y \leq y \leq Y$, it is easy to see by the hypotheses $1 + (\sinh y)^2 > (a \pm b\frac{\pi}{2})^2 + b^2y^2$. On a side of the form $z = x \pm iY$ with $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, it is easy to see that $(\sin x)^2 + (\sinh Y)^2 > (a + bx)^2 + b^2Y^2$ since for Y sufficiently large, we will have $(\sinh Y)^2 - b^2Y^2 > \sup\limits_{-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}} (a + bx)^2 - (\sin x)^2)$. □

8.4 Hurwitz's Theorem

Hurwitz's Theorem can be obtained as a corollary of Rouché's Theorem.

THEOREM 83 *Let Ω be open in $\mathbb{C}$ and let $(f_k)_{k=1}^\infty$ be a sequence of holomorphic functions converging uniformly on compacta to a (necessarily holomorphic) function f. Let $\zeta \in \Omega$ and suppose that f has a zero of finite order m at ζ. Let $\epsilon > 0$. Then there exists $\rho > 0$ and $K \in \mathbb{N}$ such that*

- $\rho < \epsilon$.
- $\{z; |z - \zeta| < \rho\} \subseteq \Omega$
- *For $k \geq K$, f_k has exactly m zeros in $\{z; |z - \zeta| < \rho\}$.*

Proof. We write $f(z) = (z-\zeta)^m g(z)$ where g is holomorphic near ζ and $g(\zeta) \neq 0$. Now choose $\rho > 0$ smaller than ϵ and such that $|z - \zeta| < \rho \Longrightarrow |g(z)| > \frac{1}{2}|g(\zeta)|$. Such a ρ exists by the continuity of g at ζ. Now choose K such that

$$k \geq K \Longrightarrow \sup_{|z-\zeta|=\rho} |f(z) - f_k(z)| < \tfrac{1}{2}\rho^m |g(\zeta)| < \inf_{|z-\zeta|=\rho} |f(z)|$$

Applying Rouché's Theorem now yields that f and f_k have the same number of zeros in $\{z; |z - \zeta| < \rho\}$. Hence the result. ■

COROLLARY 84 *Let Ω be connected open in $\mathbb{C}$ and let $(f_k)_{k=1}^\infty$ be a sequence of one-to-one holomorphic functions converging uniformly on compacta to a (necessarily holomorphic) function f. The either f is constant or one-to-one.*

Proof. Suppose that f is not one-to-one. Then There exist $z_1 \neq z_2$ in Ω such that $f(z_1) = f(z_2)$. Subtracting $f(z_1)$ from both f and f_k we can assume without loss of generality that $f(z_1) = f(z_2) = 0$. Now if either z_1 or z_2 is a zero of infinite order, then f is identically zero. We can therefore assume that they are zeros of finite order (≥ 1). Choose the ϵ of Hurwitz's Theorem to be half the distance from z_1 to z_2. Then we obtain two disjoint disks D_1 and D_2 centred at z_1 and z_2 respectively and such that for k sufficiently large, f_k has at least one zero in both D_1 and D_2. But this implies that f_k is not one-to-one contrary to hypothesis. ■

There is of course an inverse function theorem for suitably differentiable functions defined on subsets of $\mathbb{R}^d$. The following version is particular to holomorphic functions and the proof uses the ideas of this section.

THEOREM 85 (INVERSE FUNCTION THEOREM FOR HOLOMORPHIC FUNCTIONS) *Let Ω be open in $\mathbb{C}$, $\zeta \in \Omega$ and $f : \Omega \longrightarrow \mathbb{C}$ be holomorphic. Suppose that $f'(\zeta) \neq 0$. Then there is a neighbourhood U of ζ, a neighbourhood V of $f(\zeta)$ such that f maps U onto V bijectively. Furthermore the inverse function $g : V \longrightarrow U \subseteq \mathbb{C}$ is holomorphic.*

Proof. By subtracting $f(\zeta)$ from f we can assume without loss of generality that $f(\zeta) = 0$, and in view of $f'(\zeta) \neq 0$, ζ is a single zero of f. As in the proof of Hurwitz's Theorem, we can find $\rho > 0$ such that $\kappa = \inf_{|z-\zeta|=\rho} |f(z)| > 0$. Let $V = \{w; |w| < \frac{1}{2}\kappa\}$. Then V is certainly open in $\mathbb{C}$. For $w \in V$, the function $z \mapsto f(z) - w$ has a single zero in $\{z; |z-\zeta| < \rho\}$ again as in the proof of Hurwitz's Theorem. Let this single zero be designated $g(w)$. Then $U = \{z; |z-\zeta| < \rho\} \cap f^{-1}(V)$ is open in Ω and f is one-to-one on U.

Now fix $w \in V$ and consider

$$\frac{1}{2\pi i} \int_\Gamma \frac{z f'(z)}{f(z) - w} dz \tag{8.1}$$

where Γ is the circle $|z - \zeta| = \rho$ traversed anticlockwise. The integrand has a single simple pole at $z = g(w)$ and we compute the residue by the usual method for simple poles

$$\mathrm{Res}\left(\frac{z f'(z)}{f(z) - w}, g(w)\right) = \frac{g(w) f'(g(w))}{f'(g(w))} = g(w)$$

Therefore (8.1) evaluates to $g(w)$. To show that g is holomorphic, it suffices to differentiate under the integral sign. We obtain

$$\begin{aligned} \frac{\partial g}{\partial \overline{w}} &= 0 \\ g'(w) &= \frac{1}{2\pi i} \int_\Gamma \frac{z f'(z)}{(f(z) - w)^2} dz \end{aligned}$$

and we note that $g'(w)$ is clearly a continuous function of w for $w \in V$ since $f(z)$ for $z \in \Gamma$ and $w \in V$ are always separated by at least $\frac{1}{2}\kappa$. ■

COROLLARY 86 *Let Ω be open in $\mathbb{C}$ and f be a holomorphic function on Ω which is one-to-one. Then f' cannot vanish on Ω.*

Proof. We suppose the contrary, namely that there exists $\zeta \in \Omega$ such that $f'(\zeta) = 0$. Then by Proposition 51, we can write $f(z) = f(\zeta) + (h(z))^m$ locally near ζ where m is an integer $m \geq 2$, h is holomorphic near ζ, $h(\zeta) = 0$ and $h'(\zeta) \neq 0$. According to Theorem 85, h is bijective in a neighbourhood of ζ. Fix $\lambda = e^{2\pi i/m}$. Hence, for $r > 0$ small, there exist z_1, z_2 such that $h(z_1) = r$ and $h(z_2) = r\lambda$. Clearly $z_1 \neq z_2$. However, $f(z_1) = f(z_2)$ contradicting the fact that f is one-to-one. ■

9

Conformal Mapping

We start with the ***Schwarz Lemma*** , a simple result with far reaching consequences.

LEMMA 87 *Let f be a holomorphic mapping of the open unit disk to itself. Suppose that $f(0) = 0$. Then*

(i) $|f(z)| \leq |z|$ *for* $|z| < 1$. *Furthermore if equality* $|f(\zeta)| = |\zeta|$ *holds for a single* ζ *with* $0 < |\zeta| < 1$, *then there exists* $\omega \in \mathbb{C}$ *with* $|\omega| = 1$ *such that* $f(z) = \omega z$ *for all* $|z| < 1$.

(ii) $|f'(0)| \leq 1$. *Furthermore if equality* $|f'(0)| = 1$ *holds, then there exists* $\omega \in \mathbb{C}$ *with* $|\omega| = 1$ *such that* $f(z) = \omega z$ *for all* $|z| < 1$.

Proof. We can always write $f(z) = zg(z)$ where g is holomorphic. Let $0 < r < 1$. Then on $|z| = r$ we have $|g(z)| < \dfrac{|f(z)|}{|z|} < r^{-1}$. Therefore by the maximum principle $|g(z)| \leq r^{-1}$ for $|z| \leq r$. Let $r \to 1-$, then $|g(z)| \leq 1$ for $|z| < 1$, and the result of (i) follows.

If $|f(\zeta)| = |\zeta|$ for ζ such that $0 < |\zeta| < 1$, then $|g(\zeta)| = 1$. By Corollary 61, g is constant (it attains its maximum modulus at an interior point). Call the constant ω. Then using $|f(\zeta)| = |\zeta|$ again we see that $|\omega| = 1$. We have now $f(z) = \omega z$ as required.

The assertion (ii) follows in the same way from the Cauchy Estimate for $f'(0)$ based on the values of f on $|z| = r$ and by letting $r \to 1-$. Clearly $f'(0) = g(0)$, so the case of equality is also handled similarly. ∎

COROLLARY 88 *Let f be a bijective holomorphic mapping of the open unit disk onto itself, with $f(0) = 0$, then there exists $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that $f(z) = \omega z$ for all $|z| < 1$.*

Proof. Let h be the inverse mapping. By Corollary 86, f' does not vanish on the open unit disk. Hence, by Theorem 85, h is holomorphic and also a bijective mapping of the open unit disk onto itself. Applying the Schwarz Lemma 87, we get $|f(z)| \le |z|$ and $|h(w)| \le |w|$ for all z and w in the open unit disk. Putting $w = f(z)$, we find $|f(z)| = |z|$ for $|z| < 1$. Now let $g(z) = \dfrac{f(z)}{z}$, then g is holomorphic with constant modulus. We differentiate $|g(z)|^2 = g(z)\overline{g(z)}$ with respect to $\overline{z}$ to get

$$0 = g(z)\frac{\partial \overline{g}}{\partial \overline{z}}(z) + \overline{g(z)}\frac{\partial g}{\partial \overline{z}}(z) = g(z)\overline{g'(z)}$$

If both g and g' are not identically zero on the open unit disk, then both have a most countably many zeros and we have a contradiction. So at least one of these functions is identically zero and either way, g is constant. it follows that there exists $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that $f(z) = \omega z$ for all $|z| < 1$. ■

COROLLARY 89 *Let f be a bijective holomorphic mapping of the open unit disk onto itself, then there exists $\omega \in \mathbb{C}$ with $|\omega| = 1$ and $a \in \mathbb{C}$ with $|a| < 1$ such that*

$$f(z) = \omega \frac{z - a}{1 - \overline{a}z} \qquad \text{for all } |z| < 1.$$

Proof. Let a be the point of the open unit disk that gets mapped to 0. Then define

$$g(w) = f(\varphi(w)) \ \text{ where } \ \varphi(w) = \frac{a + w}{1 + \overline{a}w}$$

We remark that φ is a Möbius transformation preserving the open unit disk, so that g is a bijective holomorphic mapping of the open unit disk onto itself with $g(0) = f(a) = 0$. By Corollary 88, there exists $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that $g(w) = \omega w$. Put

$$w = \frac{z - a}{1 - \overline{a}z}$$

and we discover that $\varphi(w) = z$ (this is the inverse Möbius transformation) and hence

$$f(z) = g(w) = \omega w = \omega \frac{z-a}{1-\overline{a}z}.$$

■

This result tells that the group of bijective holomorphic mappings of the open unit disk onto itself (under composition) consists of the Möbius transformations that have the same property. Each Möbius transformation φ is associated with an invertible linear transformation λ of $\mathbb{C}^2$ via

$$\varphi(z) = \frac{w_1}{w_2} \text{ where } \lambda(z,1) = (w_1, w_2)$$

and we think of λ as acting on the one-dimensional subspaces of $\mathbb{C}^2$. The fact that φ preserves the open unit disk corresponds to the fact that λ preserves the lines that live in $|w_1| < |w_2|$, or what amounts to the same thing, the sign of the quadratic form $|w_2|^2 - |w_1|^2$. So, in our case, the matrix U of λ will be

$$U = \begin{pmatrix} \omega & -a\omega \\ -\overline{a} & 1 \end{pmatrix}.$$

If we let the matrix of the quadratic form $|w_2|^2 - |w_1|^2$ be

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We will have the matrix relation $U^\star J U = (1 - |a|^2)J$ or in longhand

$$\begin{pmatrix} \overline{\omega} & -a \\ -\overline{a\omega} & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \omega & -a\omega \\ -\overline{a} & 1 \end{pmatrix} = \begin{pmatrix} -(1-|a|^2) & 0 \\ 0 & (1-|a|^2) \end{pmatrix}.$$

9.1 Some Standard Conformal Maps

The really obvious ones are translations, rotations and scalings. Let's look at some of the less obvious ones. Usually, we are trying to map onto a disk or a region that we already know how to map conformally onto a disk.

EXAMPLE The halfspace. We can map the right halfspace $\{z \in \mathbb{C}; \Re z > 0\}$ onto the open unit disk by the Möbius transformation

$$f(z) = \frac{z-1}{z+1}$$

□

EXAMPLE We can map a sector with opening angle 2α (with $\alpha < \pi$) say $\{z \in \mathbb{C}; -\alpha < \arg(z) < \alpha\}$ onto the right halfspace with $f(z) = z^{\frac{\pi}{2\alpha}}$. This is the principal branch that is intended here, i.e.

$$f(re^{i\theta}) = r^{\frac{\pi}{2\alpha}} e^{i\frac{\pi\theta}{2\alpha}} \qquad r > 0, -\pi < \theta < \pi.$$

The same transformation can also be used to a sector $\{z \in \mathbb{C}; -\alpha < \arg(z) < \alpha, |z| < 1\}$ to the intersection of the unit disk and the right halfspace $\{z \in \mathbb{C}; \Re z > 0, |z| < 1\}$ □

EXAMPLE The intersection of the unit disk and the right halfspace $\{z \in \mathbb{C}; \Re z > 0, |z| < 1\}$ can be mapped conformally onto the first quadrant $\{z \in \mathbb{C}; \Re z > 0, \Im z > 0\}$ by

$$f(z) = \frac{z + i}{z - i}$$

Note that $-i$ gets mapped to 0 and i gets mapped to the point at infinity. The first quadrant is a sector so that can then be mapped conformally onto a halfspace.

More generally, the eye-shaped region between two circles can be mapped to a sector by

$$f(z) = \frac{z - \alpha}{z - \beta}$$

where α and β are the points where the circles intersect. This is a Möbius transformation, so that the angle of the resulting sector is the same as the angle between the two given circles. □

EXAMPLE A strip can be mapped to a sector by the exponential mapping. For example, the strip $\{z \in \mathbb{C}; a < \Re z < b\}$ can be mapped to a sector by $f(z) = e^{icz}$ with c real and suitably chosen.

The region given by $\{z \in \mathbb{C}; 0 < \Re z < 1, \Im z > 0\}$ can be mapped to a sector by $f(z) = e^{i\frac{\pi}{2}z}$ □

Finally lets look at examples with slits. The general strategy is: first move the slit to where you want it (the ends should be either at 0 or at ∞ or both) and second open the slit by taking a square root.

EXAMPLE Let $-1 < a < 1$. Consider the unit disc with the segment of the real axis from -1 to a removed. First use the Möbius transformation $f(z) = \dfrac{z - a}{1 - az}$ which preserves the unit disk and will reduce to the case $a = 0$. Then the principal square root will map onto $\{z \in \mathbb{C}; \Re z > 0, |z| < 1\}$. □

EXAMPLE The final example is $\mathbb{C}_\infty \setminus [0,1]$. Map first with the Möbius transformation $f(z) = \dfrac{-z}{1-z}$ which maps onto $\mathbb{C}_\infty \setminus [-\infty, 0]$. Then take the principal square root to map onto the right halfspace. □

9.2 Montel's Theorem

The objective of this section is to apply the Ascoli–Arzela Theorem in the context of spaces of holomorphic functions. There is an advanced version of the Ascoli–Arzela Theorem that applies more directly in the present context, but we will make do with the version established in MATH 354.

THEOREM 90 (ASCOLI–ARZELA THEOREM) *Let K be a compact metric space. We denote by $C(K)$ the space of bounded complex-valued continuous functions on K.*

Let $F \subseteq C(K)$. Then the following are equivalent statements.

- *F has compact closure in $C(K)$.*
- *F is bounded in $C(K)$ and F is equicontinuous.*

We start with the following Lemma.

LEMMA 91 *Let K, L be a compact subsets of $\mathbb{C}$, $\delta > 0$ such that $K + D(\delta) \subseteq L \subset \Omega \subset \mathbb{C}$ where Ω is open in $\mathbb{C}$. We have denoted $D(\delta)$ the closed disk centred at 0 of radius δ. Let f be holomorphic in Ω, then*

$$|f(z_1) - f(z_2)| \leq 4\delta^{-1}|z_1 - z_2| \sup_{z \in L} |f(z)| \qquad \text{for } z_1, z_2 \in K.$$

Proof. In case that $|z_1 - z_2| \geq \frac{1}{2}\delta$, we simply use the estimate

$$|f(z_1) - f(z_2)| \leq |f(z_1)| + |f(z_2)| \leq 2 \sup_{z \in L} |f(z)|.$$

So, we can assume that $|z_1 - z_2| \leq \frac{1}{2}\delta$. In that case, L contains the union of the two disks centred at z_1 and z_2 respectively of radius δ and therefore L also contains the disk centred at $\frac{1}{2}(z_1+z_2)$ of radius $\frac{3}{4}\delta$. We apply the Cauchy Integral formula on the boundary Γ of that disk to get

$$\begin{aligned}
f(z_1)-f(z_2) &= \frac{1}{2\pi i}\int_\Gamma \left(\frac{1}{z-z_1}-\frac{1}{z-z_1}\right) f(z)dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{(z_1-z_2)f(z)}{(z-z_1)(z-z_2)}dz \\
|f(z_1)-f(z_2)| &\le \frac{1}{2\pi}\int_\Gamma \frac{|z_1-z_2||f(z)|}{|z-z_1||z-z_2|}ds \\
&\le \frac{1}{2\pi}(2\pi\frac{3}{4}\delta)|z_1-z_2|\sup_{z\in L}|f(z)|\frac{1}{(\frac{1}{2}\delta)^2}
\end{aligned}$$

since for $z\in\Gamma$ we have $|z-z_j|\ge\frac{1}{2}\delta$ for $j=1,2$. Hence the result. ■

PROPOSITION 92 *Let Ω be an open subset of $\mathbb{C}$, F a family of holomorphic functions on Ω. Suppose that for every compact subset of Ω, the family*

$$F|_K = \{f|_K; f\in F\}$$

is a bounded subset of $C(K)$, then for every compact subset K of Ω, $F|_K$ is equicontinuous in $C(K)$.

Proof. For every $n\in\mathbb{N}$ we define

$$K_n = \{z; \mathrm{dist}_{\mathbb{C}\setminus\Omega}(z)\ge\frac{1}{n}, |z|\le n\}.$$

Then K_n is a closed bounded (and hence compact) subset of $\mathbb{C}$ contained in Ω. On the other hand, if K is a compact subset of $\mathbb{C}$ contained in Ω, then $\mathrm{dist}_{\mathbb{C}\setminus\Omega}$ attains its minimum value and $z\mapsto|z|$ attains its maximum value on K. It follows that there exists $n\in\mathbb{N}$ such that $K\subseteq K_n$. If as a shorthand we denote $F_n=F|_{K_n}$, then our hypothesis is that F_n is bounded in $C(K_n)$ for every $n\in\mathbb{N}$ and the desired conclusion is that F_n is equicontinuous in $C(K_n)$ for every $n\in\mathbb{N}$. However, $K_n+D(\frac{1}{n(n+1)})\subseteq K_{n+1}$ and it follows immediately from Lemma 91 that the boundedness of F_{n+1} in $C(K_{n+1})$ implies the equicontinuity of F_n in $C(K_n)$. ■

THEOREM 93 (MONTEL'S THEOREM) *Let Ω be an open subset of $\mathbb{C}$, F a family of holomorphic functions on Ω. Suppose that for every compact subset of Ω, the family $F|_K$ is bounded in $C(K)$. Then every sequence $(f_n)_{n=1}^\infty$ in F possesses a subsequence that converges uniformly on the compacta of Ω to a function holomorphic in Ω.*

Montel's Theorem is the Heine–Borel Theorem of complex function theory. If we let $H(\Omega)$ be the space of holomorphic functions on Ω with the convergence of uniform on compacta convergence (which can in fact be realised as a metric space), then every closed bounded subset of $H(\Omega)$ is compact.

Proof. With the same notations as in Proposition 92, we use the Ascoli–Arzela Theorem to extract a subsequence $(f_{m(1,n)})_{n=1}^\infty$ of $(f_n)_{n=1}^\infty$ with $(f_{m(1,n)}|_{K_1})_{n=1}^\infty$ converging uniformly on K_1. Then from that subsequence, a further subsequence $(f_{m(2,n)})_{n=1}^\infty$ of functions such that $(f_{m(2,n)}|_{K_2})_{n=1}^\infty$ converges uniformly on K_2. Then from that subsequence, a further subsequence $(f_{m(3,n)})_{n=1}^\infty$ of functions such that $(f_{m(3,n)}|_{K_3})_{n=1}^\infty$ converges uniformly on K_3... and so forth. Finally we see that the diagonal subsequence $(f_{m(n,n)})_{n=1}^\infty$ converges to a limit f uniformly on each K_n and hence on all the compacta of Ω. The reason for this is that the sequence $(f_{m(n,n)})_{n=k}^\infty$ is in fact a subsequence of the sequence $(f_{m(k,n)})_{n=k}^\infty$ which is known to converge uniformly on K_k. Finally, let us note that the limiting function f is holomorphic in Ω by Morera's Theorem. ■

9.3 The Riemann Mapping Theorem

THEOREM 94 (THE RIEMANN MAPPING THEOREM) *Let Ω be an open connected proper subset of $\mathbb{C}$ which is simply connected. Let $\zeta \in \Omega$. Then there is a unique bijective holomorphic map $f : \Omega \longrightarrow \{z \in \mathbb{C}; |z| < 1\}$ such that $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

The theorem stated under the additional assumption that the boundary of Ω is piecewise smooth was established by Bernhard Riemann in 1851. The first proof of the theorem in the generality given above is due to Constantin Carathéodory in 1912.

The uniqueness assertion is easy. It follows directly from Corollary 88. It is clear that the whole of $\mathbb{C}$ is not conformally equivalent to the open unit disk, since the latter has non-constant bounded holomorphic functions and the former does not.

LEMMA 95 *Let Ω be a connected simply connected open subset of $\mathbb{C}$ and $h : \Omega \longrightarrow \mathbb{C} \setminus \{0\}$ be holomorphic. Then h has a holomorphic square root.*

Proof. We have that Ω is path connected, locally path connected and simply connected. The map

$$z \mapsto k(z) = \frac{h(z)}{|h(z)|}$$

is a continuous map from Ω to $\mathbb{T}$. Therefore, by Theorem 77, there is a lift $\tilde{k}(z)$ and therefore a logarithm $z \mapsto \ell(z) = \ln(|h(z)|) + i\tilde{k}(z)$. Clearly $h(z) = \exp(\ell(z))$ and since the exponential map is locally invertible, we see that ℓ is holomorphic. A desired square root is given by $\exp(\frac{1}{2}\ell(z))$. ■

PROPOSITION 96 *It suffices to prove the Riemann mapping Theorem in the case where Ω is a connected simply connected bounded nonempty open subset of $\mathbb{C}$.*

Proof. Let $a \in \mathbb{C} \setminus \Omega$. Then by the previous lemma, there exists a holomorphic mapping $\varphi : \Omega \longrightarrow \mathbb{C}$ such that $(\varphi(z))^2 = z - a$. Then $z_1, z_2 \in \Omega, \varphi(z_1) = \pm\varphi(z_2)$ implies that $z_1 = z_2$ so that φ is one-to-one. Now φ is non-constant and therefore $\varphi(\Omega)$ is open in $\mathbb{C}$ by the Open Mapping Theorem52. There is a small disk V centred at $\varphi(\zeta)$ contained in $\varphi(\Omega)$. We will show that $-V$ is disjoint from $\varphi(\Omega)$. Indeed, if $z \in V$ and $-z = \varphi(z_1)$ with $z_1 \in \Omega$, then there also exists $z_2 \in \Omega$ such that $z = \varphi(z_2)$. But then $z_1 = z_2$, $z = 0$, $\varphi(z_1) = 0$ and $z_1 = a$. This is a contradiction since $z_1 \in \Omega$ and $a \notin \Omega$. Hence there is a whole disk disjoint from $\varphi(\Omega)$. It suffices to compose φ with a Möbius transformation to map Ω conformally onto a bounded open subset of $\mathbb{C}$. Since the resulting conformal transformation is continuous and has a continuous inverse, it is connected and simply connected (as well as being nonempty). ■

PROPOSITION 97 *Let Ω be a bounded nonempty open subset of $\mathbb{C}$ and let $\zeta \in \Omega$. Let F be the set of holomorphic functions from Ω to the open unit disk such that $f(\zeta) = 0$, $f'(\zeta) > 0$ and f is one-to-one. The F is nonempty. Let*

$$a = \sup_{f \in F} f'(\zeta) > 0.$$

Then the supremum is attained.

Proof. Taking $f(z) = \epsilon(z - \zeta)$ for $\epsilon > 0$ but sufficiently small shows that F is nonempty. Let (f_n) be a sequence of functions in F for which the sup is approached. Then by Montel's Theorem (f_n) possesses a subsequence converging uniformly on compacta to a holomorphic function f. We rename the subsequence to (f_n). Clearly $f'(\zeta) = a$ since uniform on compacta convergence will also imply uniform on compacta convergence of the derivative. Also $f(\zeta) = 0$ and f takes values in the closed unit disk. Now suppose that $z_1, z_2 \in \Omega$ and $f(z_1) = f(z_2)$, we will produce a contradiction. Let D be a closed disk centred at z_2 with $z_1 \notin D$ and $D \subset \Omega$. Then $z \mapsto f_n(z) - f_n(z_1)$ does not vanish on D. By Hurwitz's Theorem, since $f_n - f_n(z_1)$ converges uniformly on compacta to $f - f(z_1)$, either f is identically $f(z_1)$ on D or $f - f(z_1)$ never vanishes on D. But $z_2 \in D$ and so it must be that f is identically $f(z_1)$ on D and hence also on Ω. But then $f'(\zeta) = 0$ a contradiction. Hence f is one-to-one on Ω.

But now, from the Open Mapping Theorem, $f(\Omega)$ is open in $\mathbb{C}$ and hence f takes values in the open unit disk. So $f \in F$ as required. ∎

Proof of the Riemann Mapping Theorem. We can assume that Ω is bounded and with the notations of Proposition 97 let f be a function in F for which the supremum is attained. It suffices to show that f maps *onto* the open unit disk.

Let $\alpha \in \mathbb{C}$, $|\alpha| < 1$ and suppose that f does not take the value α. Then applying Lemma 95 again, there is a holomorphic function h in Ω such that

$$(h(z))^2 = \frac{f(z) - \alpha}{1 - \overline{\alpha} f(z)}. \tag{9.1}$$

Clearly h takes values in the open unit disk and is one-to-one on Ω. However, h does not necessarily map ζ to 0. We therefore define

$$g(z) = \frac{|h'(\zeta)|}{h'(\zeta)} \left(\frac{h(z) - h(\zeta)}{1 - \overline{h(\zeta)} h(z)} \right) \tag{9.2}$$

and clearly g is one-to-one holomorphic from Ω to the open unit disk and $g(\zeta) = 0$.

Differentiating (9.1) and setting $z = \zeta$ yields

$$2h(\zeta)h'(\zeta) = (1 - |\alpha|^2) f'(\zeta)$$

and processing (9.2) similarly gives

$$g'(\zeta) = \frac{|h'(\zeta)|}{h'(\zeta)} \left(\frac{(1 - |h(\zeta)|^2) h'(\zeta)}{(1 - |h(\zeta)|^2)^2} \right) = \frac{|h'(\zeta)|}{1 - |\alpha|}.$$

Therefore

$$g'(\zeta) = \frac{1-|\alpha|^2}{2\sqrt{|\alpha|}(1-|\alpha|)} f'(\zeta) = \frac{1+|\alpha|}{2\sqrt{|\alpha|}} f'(\zeta) > f'(\zeta),$$

since $|\alpha| < 1$. Hence $g \in F$ and we have a contradiction with the definition of f.

■

9.4 Conformal maps between Annuli

In this section we let $0 < r_j < 1$ for $j = 1, 2$ and let $A_j = \{z \in \mathbb{C}; r_j < |z| < 1\}$ be the corresponding annulus. We ask what conformal maps are possible from A_1 onto A_2.

THEOREM 98 *Let φ_1 be a conformal map from A_1 onto A_2, then $r_1 = r_2$ and there exists $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that either $\varphi_1(z) = \omega z$ or $\varphi_1(z) = r_1 \omega z^{-1}$.*

Sketch proof [1].

Let φ_2 be the inverse map to φ_1. Let $S_j = \{z \in \mathbb{C}; \ln(r_j) < \Re z < 0\}$ be open strips in the complex plane for $j = 1, 2$. The exponential map takes S_j onto A_j but is not one-to-one. By Theorem 77 and the fact that strips are simply connected, we can construct a continuous map $\psi_1 : S_1 \longrightarrow S_2$ such that $\varphi_1 \circ \exp = \exp \circ \psi_1$. Since locally $\psi_1 = \log \circ \varphi_1 \circ \exp$ where log is some branch of the logarithm, ψ_1 is holomorphic. Let $s_1 \in S_1$. Let $s_2 = \psi_1(s_1) \in S_2$. Similarly, we may construct a holomorphic mapping $\psi_2 : S_2 \longrightarrow S_1$ such that $\varphi_2 \circ \exp = \exp \circ \psi_2$ and additionally we may arrange that $\psi_2(s_2) = s_1$.

Clearly $\psi_2 \circ \psi_1(z) = z$ for all z near s_1 and hence for all $z \in S_1$. Similarly, $\psi_1 \circ \psi_2(z) = z$ for all $z \in S_2$. Hence ψ_1 is a conformal map from S_1 onto S_2. We know how to classify these, but before we get into that aspect of the proof, let us consider $\psi_1(z + 2\pi i) - \psi_1(z)$ which takes values in $2\pi i\mathbb{Z}$. Since a continuous mapping from a connected space to a discrete one is necessarliy constant, there is an integer m_1 such that $\psi_1(z + 2\pi i) = \psi_1(z) + 2m_1\pi i$ for all $z \in S_1$. Similarly, there exists $m_2 \in \mathbb{Z}$ such that $\psi_2(z + 2\pi i) = \psi_2(z) + 2m_2\pi i$ for all $z \in S_2$. We now have (with some details omitted)

$$\begin{aligned} z + 2\pi i = \psi_2 \circ \psi_1(z + 2\pi i) = \psi_2(\psi_1(z) + 2m_1\pi i) = \psi_2 \circ \psi_1(z) + 2m_1 m_2 \pi i \\ = z + 2m_1 m_2 \pi i \end{aligned}$$

[1] With thanks to Paul Koosis for suggesting this approach.

forcing $m_1 m_2 = 1$ and therefore $m_1 = m_2 = \pm 1$.

Let $H = \{z \in \mathbb{C}; \Im z > 0\}$, the upper halfspace. The conformal mapping $\chi_j : S_j \longrightarrow H$ is given by

$$\chi_j(z) = \exp\left(\frac{i\pi z}{\ln(r_j)}\right).$$

Defining $\mu_1 = \chi_2 \circ \psi_1 \circ \chi_1^{-1}$ we have a conformal map μ_1 of H onto H. We define $t_j = \exp\left(-\dfrac{2\pi^2}{\ln(r_j)}\right) > 1$, so that $\chi_j(z + 2\pi i) = t_j \chi_j(z)$ and it follows that $\mu_1(t_1 w) = t_2^{m_1} \mu_1(w)$ for all $w \in H$. Now by Corollary 89 and the standard conformal equivalence between disk and halfspace, we see that μ_1 has the form

$$\mu_1(w) = \frac{aw + b}{cw + d}$$

with a, b, c, d real and $ad - bc > 0$. We now obtain the identity

$$\frac{at_1 w + b}{ct_1 w + d} = \mu_1(t_1 w) = t_2^{m_1} \mu_1(w) = t_2^{m_1} \frac{aw + b}{cw + d}.$$

We multiply out and equate the coefficients of 1, w and w^2 to get $ac = t_2^{m_1} ac$, $adt_1 + bc = t_2^{m_1}(ad + bct_1)$ and $bd = t_2^{m_1} bd$. So, either $a = 0$ or $c = 0$ and either $b = 0$ or $d = 0$. Two of the resulting four cases violate $ad - bc > 0$. The remaining two are

- $b = c = 0$, $t_1 = t_2^{m_1}$ and since $t_1, t_2 > 1$, $m_1 = 1$, $t_1 = t_2$, $\mu_1(w) = \alpha w$ with $\alpha > 0$.
- $a = d = 0$, $t_1 = t_2^{-m_1}$ and since $t_1, t_2 > 1$, $m_1 = -1$, $t_1 = t_2$, $\mu_1(w) = -\alpha w^{-1}$ with $\alpha > 0$.

Tracing this information back and using the fact that ψ_1 preserves S_1 (now known to equal S_2) shows that ψ_1 assumes one or other of the forms

$$\psi_1(z) = z + i\beta \qquad \text{or} \qquad \psi_1(z) = \ln(r_1) - z + i\beta,$$

with $\beta \in \mathbb{R}$ and finally tracing back to φ_1 establishes the result. ■

In the same vein, we can show the following theorem.

THEOREM 99 *Let ℓ, u be continuous functions from $\mathbb{T}$ to $]0, \infty[$ such that $\ell < u$ everywhere. Let Ω be the open subset $\{re^{i\theta}; \ell(e^{i\theta}) < r < u(e^{i\theta})\}$. Then there exists $s \in]0, 1[$ such that Ω is conformally equivalent to the annulus $\{z \in \mathbb{C}; s < |z| < 1\}$.*

Sketch proof. The proof follows a very similar line to the proof of Theorem 98. Let $U = \{z; \exp(z) \in \Omega\}$ and $V = \{\exp(iaz); z \in U\}$ where $a > 0$ is chosen sufficiently small that $a(\ln(\sup u) - \ln(\inf \ell)) < \pi$ which forces V to lie in a halfspace. Now clearly, U is contractible and so is V, so by the Riemann Mapping Theorem, V is conformally equivalent to the upper halfspace H. There is some freedom for the choice of this mapping which we will exploit later. Let $\varphi : V \longrightarrow H$ be such a conformal equivalence. Now $z \in U \iff z + 2\pi i \in U$, so $z \in V \iff e^{-2\pi a} z \in V$. Since the only conformal transformations of H onto H are Möbius transformations, we have

$$\varphi(e^{-2\pi a} \varphi^{-1}(z)) = \frac{az + b}{cz + d}$$

for all $z \in H$ and where a, b, c, d are real and $ad - bc > 0$. Equivalently

$$\varphi(e^{-2\pi a} z) = \frac{a\varphi(z) + b}{c\varphi(z) + d} \tag{9.3}$$

for all $z \in V$. We now exploit the freedom in the choice of the mapping φ. This allows us to replace the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with the matrix

$$S^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} S$$

where S is a matrix with real entries and positive determinant. Therefore, using the Jordan canonical form for real matrices, we can are arrange that the matrix A has one or other of the special forms

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \lambda, \mu \in \mathbb{R}, \lambda\mu > 0$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \lambda \in \mathbb{R} \setminus \{0\}$$

$$\begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix}, \mu, \nu \in \mathbb{R}, \nu \neq 0$$

Thus (9.3) can always be rewritten in one or other of the following forms

(i) $\varphi(e^{-2\pi a}z) = \lambda\varphi(z)$ with $\lambda > 0$.

(ii) $\varphi(e^{-2\pi a}z) = \varphi(z) + \mu$ with $\mu \in \mathbb{R} \setminus \{0\}$.

(iii) $\varphi(e^{-2\pi a}z) = \dfrac{\varphi(z) - \nu}{\nu\varphi(z) + 1}$ with $\nu \in \mathbb{R} \setminus \{0\}$.

We can eliminate case (iii) immediately. Let $z \in V$ be the point such that $\varphi(z) = i$. Then

$$\varphi(e^{-2\pi a}z) = \frac{i - \nu}{\nu i + 1} = i,$$

and since φ is one-to-one, $e^{-2\pi a}z = z$ resulting in $z = 0$ which is not correct since $0 \notin V$.

Next we consider case (ii). We define $\psi(z) = 2\pi i\mu^{-1}\varphi(\exp(iaz))$ and it can be shown that ψ is a conformal map of U onto the right or left halfspace (according as $\mu > 0$ or $\mu < 0$) and it satisfies $\psi(z + 2\pi i) = \psi(z) + 2\pi i$. It therefore respects the exponential mapping and factors down onto a conformal map of Ω onto either $\{z \in \mathbb{C}; 0 < |z| < 1\}$ or $\{z \in \mathbb{C}; |z| > 1\}$. In either case we find that Ω is conformally equivalent to a punctured disk and it is easily seen that this is not the case.

In case (i), we first observe that $\lambda \neq 1$ for else φ is not one-to-one. Let us assume that $\lambda < 1$, then we choose $b > 0$ such that $-\pi b^{-1} = \frac{1}{2}\ln(\lambda)$ and define ψ by the relation

$$\varphi(\exp(iaz)) = \exp(ib\psi(z))$$

The ψ is a conformal map of U onto the strip $-\pi b^{-1} < \Re z < 0$. Furthermore $\psi(z + 2\pi i) = \psi(z) + 2\pi i$. Again ψ respects the exponential map and factors down onto a conformal map of Ω onto the annulus $\{z \in \mathbb{C}; s < |z| < 1\}$ where $\ln(s) = -\pi b^{-1}$ (i.e. $s = \sqrt{\lambda}$). The case $\lambda > 1$ is similar, but it is necessary to flip the boundaries of the annulus. ■

10

Odds and Ends

10.1 The Schwarz Reflection Principle

The Schwarz Reflection Principle addresses the question of analytic continuation. That's the situation in which a holomorphic function in one domain is extended to a holomorphic function in a larger domain. We have seen instances of this already in these notes and it is a very common theme in complex analysis. A systematic treatment however is beyond the scope of this course.

Here we present just one theorem in this vein.

THEOREM 100 *Let Ω be an open subset of $\mathbb{C}$ with the property that $z \in \Omega \iff \overline{z} \in \Omega$. Let f be defined on $\{z \in \Omega; \Im z \geq 0\}$ and be continuous on that set, as well as being holomorphic in $\{z \in \Omega; \Im z > 0\}$. Suppose that $z \in \Omega$, z real implies that $f(z)$ is real. Then f extends to a holomorphic function $\tilde{f} : \Omega \longrightarrow \mathbb{C}$.*

Proof. We define

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \in \Omega, \Im z \geq 0, \\ \overline{f(\overline{z})} & \text{if } z \in \Omega, \Im z \leq 0. \end{cases}$$

The two definitions agree by hypothesis. We see that $\tilde{f}$ is a continuous map from Ω to $\mathbb{C}$ by the Glueing Lemma, the subsets $\{z \in \Omega; \Im z \geq 0\}$ and $\{z \in \Omega; \Im z \leq 0\}$ being closed subsets in the relative topology of Ω.

It remains to show that $\tilde{f}$ is holomorphic. We do this using Morera's Theorem. We need to show that if T is a solid triangle contained in Ω, then $\displaystyle\int_{\partial T} f(z)dz = 0$.

This is obvious (by Cauchy's Theorem) if T is contained in either of the sets $\{z \in \Omega; \Im z > 0\}$ and $\{z \in \Omega; \Im z < 0\}$, but is not immediately obvious in the case that the triangle straddles the real axis, or in fact even touches it. In this situation, the idea is to write the triangular path ∂T as the sum of two closed polygonal paths P_1 and P_2 with P_1 in the upper half space $\Im z \geq 0$ and P_2 in the lower half space $\Im z \leq 0$. It will be enough to show that $\int_{P_j} f(z)dz = 0$ for $j = 1, 2$.

To see this for $j = 1$, we observe by Cauchy's Theorem that $\int_{P_j + i\lambda} f(z)dz = 0$ for $\lambda > 0$ since now the translated path $P_j + i\lambda$ lives in the strict upper half space $\Im z > 0$ where f is known to be analytic. But

$$\int_{P_j + i\lambda} f(z)dz = \int_{P_j} f(z + i\lambda)dz \longrightarrow \int_{P_j} f(z)dz$$

as $\lambda \downarrow 0$, since the values of z under consideration form a closed bounded subset and f is uniformly continuous on such a subset. It follows that $\int_{P_1} f(z)dz = 0$ and the case $j = 2$ is exactly similar. ■

10.2 The Gamma Function

The Gamma Function Γ is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t} = \int_0^\infty e^{-t} t^{z-1} dt$$

for $\Re z > 0$. The integral always converges at ∞. The condition $\Re z > 0$ is imposed to make it converge at $t = 0$. In the right halfplane Γ is a uniform on compacta limit of $\int_{n^{-1}}^{n} e^{-t} t^{z-1} dt$ as $n \to \infty$, so Γ is holomorphic in the right halfplane.

It is easy to prove by integration by parts and by induction that for $n \in \mathbb{N}$ and $x > 0$

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} dt = \frac{n^x \; n!}{x(x+1)(x+2)\cdots(x+n)}$$

We claim that

$$e^{-t}\left(1 - \frac{2t^2}{n}\right) \leq \left(1 - \frac{t}{n}\right)^n \leq e^{-t} \tag{10.1}$$

for $n \in \mathbb{N}$ and $0 \le t \le n$. The right-hand inequality boils down to $1 - x \le e^x$ for $0 \le x \le 1$. The left-hand one is more subtle. Taking logs, we need

$$g_n(t) = n \ln\left(1 - \frac{t}{n}\right) + t - \ln\left(1 - \frac{2t^2}{n}\right) \ge 0$$

for $0 \le t < \sqrt{\dfrac{n}{2}}$, the inequality being obvious otherwise. Clearly $g_n(0) = 0$, so it will suffice to show that $g_n'(t) \ge 0$. We have

$$g_n'(t) = -\frac{t}{n-t} + \frac{4t}{n-2t^2} = \frac{t(3n - 4t + 2t^2)}{(n-t)(n-2t^2)} \ge 0$$

in the desired range since $3n - 4t > 3n - 2\sqrt{2}\sqrt{n} > 2\sqrt{2}(n - \sqrt{n}) \ge 0$. From(10.1) it follows that

$$\left|\Gamma(x) - \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} dt\right| \le \int_0^n \frac{2t^2}{n} e^{-t} t^{x-1} dt + \int_n^\infty e^{-t} t^{x-1} dt \underset{n\to\infty}{\longrightarrow} 0$$

for $x > 0$. It follows that

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x \; n!}{x(x+1)(x+2)\cdots(x+n)},$$

again for $x > 0$.

We set about defining $\dfrac{1}{\Gamma(z)}$ for all complex z by means of

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right) \tag{10.2}$$

where γ is Euler's constant. For $|w| \le \frac{1}{2}$ one has

$$|\log(1+w) - w| \le \left|\sum_{k=2}^{\infty} (-1)^{k-1} \frac{1}{k} w^k\right| \le \frac{1}{4}|w|^2$$

and consequently

$$\left|\log(1 + \frac{z}{n}) - \frac{z}{n}\right| \le \frac{|z|^2}{4n^2}$$

for $n \geq 2|z|$. Thus, the product (10.2) converges uniformly on compact of $\mathbb{C}$ to an entire function. We find for $x > 0$

$$xe^{\gamma x} \prod_{n=1}^{N} \left(\left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}} \right) = \frac{x(x+1)(x+2)\cdots(x+N)}{N^x \; N!} e^{x\left(\gamma + \ln(N) - \sum_{n=1}^{N} \frac{1}{n}\right)}$$

with the consequence that the two definitions of Γ are both holomorphic and agree on the positive real axis. Therefore they agree everywhere on $\Re z > 0$. We note that $\dfrac{1}{\Gamma(z)}$ has zeros only at $0, -1, -2, \ldots$ and these are therefore the only poles of Γ. This is an example of analytic continuation. A function defined intitially only in a halfspace turns out to have a holomorphic extension to a much larger region.

Next consider

$$\begin{aligned} \frac{\Gamma(z+1)}{z} &= \lim_{n\to\infty} \frac{n^z \; (n+1)!}{z(z+1)(z+2)\cdots(z+n+1)} \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)\left(\frac{n^{z+1} \; n!}{z(z+1)(z+2)\cdots(z+n+1)}\right) = \Gamma(z) \end{aligned}$$

So that $\Gamma(z+1) = z\Gamma(z)$ (except where Γ has its poles).

From (10.2) we also have

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

or equivalently

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \tag{10.3}$$

It turns out that we can relate the infinite product on the right of (10.3) to the sin function.

We will need the following lemma

LEMMA 101 *Let $a_{n,k} \in \mathbb{C}$ and suppose that $\lim_{k\to\infty} a_{n,k} = a_n$ for all $n \in \mathbb{N}$. Suppose further that $|a_{n,k}| \leq M_n$ for all n and k and that $\sum_{n=1}^{\infty} M_n < \infty$. Then* *(i)* $\lim_{k\to\infty} \sum_{n=1}^{\infty} a_{n,k} = \sum_{n=1}^{\infty} a_n.$

(ii) $\lim_{k\to\infty} \prod_{n=1}^{\infty} \Big(1 + a_{n,k}\Big) = \prod_{n=1}^{\infty} \Big(1 + a_n\Big).$

Proof. This is Analysis 2 stuff, well almost. Let $\epsilon > 0$ Then since

$$\left| \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{n,k} - \sum_{n=1}^{\infty} a_n \right| \le \sum_{n=1}^{N} |a_{n,k} - a_n| + 2 \sum_{n>N} M_n$$

we first choose N so large that $\sum_{n>N} M_n < \frac{1}{4}\epsilon$ and then K so large that

$$|a_{n,k} - a_n| < \frac{1}{2N}\epsilon$$

holds for $n = 1, 2, \ldots, N$ and $k > K$. We obtain

$$\left| \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{n,k} - \sum_{n=1}^{\infty} a_n \right| < \epsilon$$

for $k > K$. This proves assertion (i).

For assertion (ii) we first choose N so large that $M_n < \frac{1}{2}$ for $n > N$. It will be enough to show that

$$\lim_{k\to\infty} \prod_{n=N+1}^{\infty} \Big(1 + a_{n,k}\Big) = \prod_{n=N+1}^{\infty} \Big(1 + a_n\Big).$$

Equivalently, we can assume without loss of generality that $M_n < \frac{1}{2}$ for all n. But now, using the principal branch of the logarithm we have

$$|\log(1 + a_{n,k})| \le 2|a_{n,k}| \le 2M_n$$

using $|\log(1 + z)| \le 2|z|$ for $|z| < \frac{1}{2}$, and the result follows from (i) since exp is continuous. ∎

PROPOSITION 102 *We have*

$$\pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)=\sin(\pi z) \tag{10.4}$$

for all $z \in \mathbb{C}$.

Proof. Let $m = 2\ell$ be an even integer. Consider the polynomial

$$P_m(z)=\frac{1}{2i}\left(\left(1+\frac{\pi i z}{m}\right)^m-\left(1-\frac{\pi i z}{m}\right)^m\right).$$

and note that it has degree $m-1$, since the term in z^m will cancel when the brackets are expanded, but the term in z^{m-1} will not. Since $\lim_{m\to\infty} m\log(1+\dfrac{\pi i z}{m})=\pi i$ (the left hand side is the derivative of $z\mapsto \log(1+\pi i z)$ at $z=0$), it follows that

$$\lim_{m\to\infty} mP_m(z)=\sin(\pi z).$$

(As an exercise, show that this is a uniform on compacta limit). On the other hand, the roots of P_m satisfy

$$\left(\frac{m+\pi i z}{m-\pi i z}\right)^m=1,$$

and hence have the form

$$\frac{m+\pi i z}{m-\pi i z}=e^{2\pi i k m^{-1}}$$

for $k=0,1,2,\ldots,m-1$. Solving for z gives

$$z=\frac{m(e^{2\pi i k m^{-1}}-1)}{\pi i(e^{2\pi i k m^{-1}}-1)}=\frac{m}{\pi}\tan\left(\frac{k\pi}{m}\right).$$

We now realise that $k=\ell$ does not yield a root and that we have our full complement of $m-1$ roots. We can therefore write

$$P_m(z)=Cz\prod_{k=1}^{\ell-1}\left(1-\frac{\pi^2 z^2}{m^2(\tan(k\pi/m))^2}\right)$$

for suitable C. From the coefficient of z obtained from the binomial expansion, we see that $C = \pi$. Hence

$$P_m(z) = \pi z \prod_{k=1}^{\ell-1} \left(1 - \frac{\pi^2 z^2}{m^2(\tan(k\pi/m))^2}\right)$$

We now apply Lemma 101 with

$$\begin{aligned} a_{n,k} &= \begin{cases} -\dfrac{\pi^2 z^2}{4k^2(\tan(n\pi/2k))^2} & \text{if } n < k, \\ 0 & \text{if } n \geq k. \end{cases} \\ a_n &= -\frac{z^2}{n^2} \\ M_n &= \frac{|z|^2}{n^2} \end{aligned}$$

to show that (10.4) holds.

COROLLARY 103 *We have*

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin(\pi z)}{\pi}$$

for all $z \in \mathbb{C}$.

One can also point out that since $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt > 0$, it must be that $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$. Putting $t = \frac{1}{2}s^2$ in the integral, this provides confirmation that

$$\int_0^\infty e^{-\frac{1}{2}s^2} ds = \sqrt{\frac{\pi}{2}}.$$

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