1. (a) Solve the Volterra integral equation $y(x) = x + \int_0^x y(\xi)\, d\xi$.

   (b) Use the Fourier transform to solve the integral equation

$$y(x) = e^{-|x|} + \lambda \int_{-\infty}^{\infty} e^{-|x-\xi|} y(\xi)\, d\xi,$$

   where $y(x)$ is to remain finite as $x \to \pm\infty$ and $\lambda < \frac{1}{2}$.

2. Employ a Green's function to transform the following boundary-value problem into an integral equation

$$(1-x)y'' - y' + \lambda x y = 0, \qquad y(0) = y'(1) = 0.$$

3. (a) Find the eigenfunction(s) and eigenvalue(s) of the integral equation

$$y(x) = \lambda \int_0^1 e^{x-\xi} y(\xi)\, d\xi.$$

   (b) Solve the integral equation

$$y(x) = e^x + \lambda \int_0^1 e^{x-\xi} y(\xi)\, d\xi$$

4. Use the Laplace transform to solve the Neumann problem for the heat-equation in a semi-infinite domain when there is an inward flux of heat at $x = 0$; i.e., solve

$$u_t = k u_{xx}, \quad 0 < x < \infty,\ t > 0$$

   subject to the initial and boundary conditions $u(x,0) = 0,\ u_x(0,t) = -q(t)$ and $\lim_{x\to\infty} u = 0$.

   Express your answer in the form

$$u(x,t) = \int_0^t g(t-\tau) f(x,\tau)\, d\tau$$

   given that $\mathcal{L}^{-1}\{p^{-1/2} e^{-x\sqrt{p/k}}\} = (\pi t)^{-1/2}\, e^{-x^2/4kt}$.

5. In a two-dimensional problem involving forced wave motion, the dependent variable $\phi(x,y,t) = u(x,y)e^{-i\omega t}$, where u satisfies the Helmholtz equation with forcing

$$\nabla^2 u + k^2 u = -\frac{g(\vec{x})}{c^2}$$

   and $k^2 = \omega^2/c^2$. Suppose that the region of interest is the half-plane $x > 0$, $g \to 0$ rapidly as $x^2 + y^2 \to \infty$ and $u_x = f(y)$ on the boundary $x = 0$.

   (a) Write down the solution $u(x,y)$ in terms of integrals involving $G(x,y;\xi,\eta)$, $f(y)$ and the non-homogeneous term in the equation for u.

   (b) What is the Green's function $G(x,y;\xi,\eta)$ and what is the boundary condition satisfied by G on $x = 0$?

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-585B

INTEGRAL EQUATIONS & TRANSFORMS

Examiner: Professor S.A. Maslowe
Associate Examiner: L.J. Campbell

Date: Friday, April 23, 1999
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

This exam comprises the cover and 1 page of questions.