1. Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$F(x, y, z) = (x^2 - y^2, xy, xz, yz), \quad (x, y, z) \in \mathbb{R}^3.$$ 

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin. Show that the map $\varphi = F\big|_{S^2}$ induces a well-defined map $\tilde{\varphi} : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^4$. Prove that $\tilde{\varphi}$ is an immersion. Is it an embedding? Justify all your answers carefully.

2. Use the Mayer-Vietoris sequence to compute the de Rham cohomology of $\mathbb{R}^2 \setminus \{(0, 0), (0, 1)\}$.

3. Compute the de Rham cohomology of the Lie group $SL(2, \mathbb{R})$.

4. Consider the Poincaré upper half-plane $(M, g)$, where $M = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ and

$$g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right) = g \left( \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right) = \frac{1}{y^2}, \quad g \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = 0.$$ 

Putting $z = x + iy$, show that the transformation $f$ given by $z \mapsto \frac{az + b}{cz + d}$, $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ is a global isometry of $(M, g)$, that is a diffeomorphism of $M$ satisfying

$$f^*g = g.$$ 

5. Consider again the Poincaré upper half-plane $(M, g)$, as in problem 4. Consider the map $\gamma : (a, b) \to M$, $a > 0$, $t \mapsto (0, t)$. Show that the image of $\gamma$ can be parametrized as a geodesic curve. Use the isometries determined in problem 4 to transform the image of $\gamma$ into parts of semi-circles. Are these also images of geodesic curves? Please justify your answer.

(Remark: These semi-circles are called horocycles.)
McGILL UNIVERSITY
FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-577B
GEOMETRY AND TOPOLOGY II

Examiner: Professor N. Kamran
Associate Examiner: Professor P. Russell

Date: Wednesday, April 19, 2000
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Calculators are not permitted.

This exam comprises the cover and one page of questions.