McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-355B

ANALYSIS IV

Examiner: Professor J.R. Choksi
Associate Examiner: Professor S.W. Drury

Date: Friday, April 24, 1998
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

CALCULATORS NOT PERMITTED.
Attempt any six (6) questions.
All questions carry equal marks.

This exam comprises the cover and 2 pages of questions.

1. For any subset $E \subseteq \mathbb{R}$ and $a \in \mathbb{R}$, let $E + a = \{x + a : x \in E\}$. Let m^* denote Lebesgue outer measure, and m Lebesgue measure. Prove that

   (a) $E + a$ is Borel if and only if E is Borel.

   (b) $m^*(E + a) = m^*(E)$, all $E \subseteq \mathbb{R}$, $a \in \mathbb{R}$.

   (c) $E + a$ is Lebesgue measurable if and only if E is Lebesgue measurable, and then $m(E + a) = m(E)$.

2. Let (X, S, μ) be a measure space and $\{f_n\}$ $n \in \mathbb{N}$, a sequence of non-negative measurable functions on (X, S, μ). State the monotone convergence theorem and Fatou's lemma for the sequence $\{f_n\}$. Assuming the monotone convergence theorem, prove Fatou's lemma. Using Fatou's lemma prove the following:

   If f, f_n, $n \in \mathbb{N}$ are non-negative integrable functions on (X, S, μ) such that $f_n \to f$ a.e. and $\int_X f_n d\mu \to \int_X f d\mu$, prove that for all $E \in S$, $\int_E f_n d\mu \to \int_E f d\mu$.

3. Compute $\lim_{n\to\infty} \int_0^1 \frac{1 + nx^2}{(1 + x^2)^n} dx$, justifying any interchanges in orders of limits.

4. Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be finite measure spaces, λ the product measure on $\mathcal{A} \otimes \mathcal{B}$. Show that the set of functions of the form

$$\sum_{j=1}^{n} f_j(x) g_j(y), \qquad f_j \in L^2(\mu),\ g_j \in L^2(\nu)\ j = 1, \cdots, n,\ n \in \mathbb{N},$$

   is dense in $L^2(\lambda)$.

   [Hint: First show that characteristic functions of sets in $\mathcal{A} \otimes \mathcal{B}$ can be approximated by such functions.]

5. Let $\{f_n\}$, $\{g_n\}$ be two complete orthonormal sequences in $L^2([0,1]$, one dimensional Lebesgue measure). Show that the set of functions

$$\{h_{n,m}(x,y) = f_m(x)g_n(y) : n, m \in \mathbb{N}\}$$

is a complete orthonormal sequence in $L^2([0,1] \times [0,1]$, two-dimensional Lebesgue measure).

[Hint: You may use question 4 above, even if you have not done it!]

6. Let $\{n_k\}$ be an increasing sequence of positive integers and E the (Lebesgue measurable) set of all x in $(-\pi, \pi)$ such that the sequence $\{\sin n_k x\}$ converges. Show that $m(E) = 0$ where m is Lebesgue measure.

[Hint: If A is any measurable subset of E, then $\int_A \sin n_k x dx \to 0$ as $k \to \infty$, but $\int_A (\sin n_k x)^2 dx \to \frac{1}{2} m(A)$.]

7. Prove or disprove the following. (i.e. if the statement is true, give a proof, if it is false give a counterexample.)

   (a) Every non-empty Borel set in $\mathbb{R}$ is either (a) an at most countable union of non-degenerate intervals or (b) an at most countable union of sets consisting of one point or (c) a finite or countable union of sets of types (a) and (b).

   (b) If $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces, λ is the product measure on $\mathcal{A} \otimes \mathcal{B}$, $f(x,y)$ is measurable λ and the repeated integral

$$\int_X \left\{ \int_Y |f(x,y)| \nu(dy) \right\} \mu(dx) < \infty,$$

   then f is integrable λ.

   (c) If $\{f_n\}$, $n \in \mathbb{N}$ is an orthonormal sequence in $L^2(X, S, \mu)$ and for all $f \in L^2$, $\|f\|^2 = \sum_{n=1}^{\infty} |(f, f_n)|^2$ then $\{f_n\}$ is complete in L^2.