MCGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-355B

ANALYSIS II (PART II)

Examiner: Professor V. Havin
Associate Examiner: Professor J.R. Choksi

Date: Friday, April 19, 1996
Time: 2:00 - 5:00 p.m.

Instructions: Solve all problems

This examination paper comprises this cover and 2 pages of questions
Mathematics Final Exam 189-355B Friday, April 19, 1996

Marks:

(10) 1. Suppose \( \nu \) is a shift invariant measure, \( \text{dom} \nu = \mathcal{A}_d \) (= the Lebesgue \( \sigma \)-algebra in \( \mathbb{R}^d \)), \( \nu([0,1] \times \cdots \times [0,1]) = 1 \). Prove that \( \nu = m_d \) (= Lebesgue measure in \( \mathbb{R}^d \)).

(10) 2. Suppose \( A \in \mathcal{A}_1, A \subset [0,1], \ 0 < y < m_1(A) \). Prove that there is a Lebesgue measurable \( B \subset A \) such that \( m_1(B) = y \). (Consider the function \( x \mapsto m_1(A \cap [0,x]) \), \( 0 \leq x \leq 1 \)).

(10) 3. Suppose \( E_1, E_2, E_3 \) are Lebesgue measurable subsets of \([0,1]\), and any \( x \in [0,1] \) belongs to at least two of \( E_j \). Prove that one of \( E_j \) satisfies the inequality \( m_1(E_j) \geq \frac{2}{3} \). (Look at \( \chi_{E_j} \)).

(20) 4. Let \( f \) be a function continuous on \([a,b]\) and differentiable at any point of \((a,b)\). Prove that if \( f' \) is bounded, then \( f(b) - f(a) = \int_a^b f' \, dm_1 \).

(10) 5. Let \((X, \mathcal{A}, \mu)\) be a measure space, \( \mu(X) < +\infty, (f_n)_{n=1}^{\infty} \) a sequence in \( L^0(X,\mu) \) (= a.e. finite measurable functions), \( f \in L^0(X,\mu) \). Prove that the following are equivalent:

\((A)\) \( f_n \xrightarrow{\text{a.e.}} f \) (B) any subsequence \((f_{n_k})_{k=1}^{\infty} (n_1 < n_2 < \cdots)\) contains a subsequence \((f_{n_{k_t}})_{t=1}^{\infty} (k_1 < k_2 < \cdots)\) such that \( f_{n_{k_t}} \rightarrow f \) a.e.

(10) 6. Suppose \( \mu(X) < +\infty, p \geq 1, (f_n)_{n=1}^{\infty} \) is a sequence in \( L^p(X,\mu), \sum_{n=1}^{\infty} \| f_n \|_p < +\infty \). Prove that \( \sum_{n=1}^{\infty} f_n(x) \) absolutely converges a.e., \( \sum_{n=1}^{\infty} f_n \in L^p(X,\mu) \), and

\[ \| \sum_{n=1}^{\infty} f_n \|_p \leq \sum_{n=1}^{\infty} \| f_n \|_p. \]
7. Let \( (f_n)_{k=1}^{\infty} \) be a sequence of non-negative functions, \( f_n \in L^1(X, \mu)(n = 1, 2, \ldots), f \in L^1(X, \mu) \).
Prove that if \( f_n \xrightarrow{n \to \infty} f \) a.e. and \( \int_X f_n \, d\mu \xrightarrow{n \to \infty} \int_X f \, d\mu \), then \( \|f_n - f\|_1 \xrightarrow{n \to \infty} 0 \). (Hint: \( (f - f_n)_+ \leq f \)). Is it true if we drop the non-negativity assumption?

8. Suppose \( f, g \in L^1(X, \mu) \). Put \( F(x, y) := f(x)g(y)(x, y \in X) \). Prove that \( F \in L^1(X \times X, \mu \otimes \mu) \), and
\[
\int_{X \times X} F \, d(\mu \otimes \mu) = \int_X f \, d\mu \cdot \int_X g \, d\mu.
\]

9. Suppose \( f_n \in L^1(X, \mu)(n = 1, \ldots), \sup_n |f_n| \in L^1(X, \mu) \). Prove that the following are equivalent:
\[(A) \quad f_n \xrightarrow{n \to \infty} 0 \quad ; \quad (B) \quad \|f_n\|_1 \xrightarrow{n \to \infty} 0.\]