McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 355

Analysis 3

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Associate Examiner: Professor S. W. Drury

Date: Wednesday April 16, 2003
Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Each question is worth 10 points.
Calculators are not permitted.
Please show all your work.

This exam comprises the cover and 1 pages of 5 questions.

Mathematics 355 : Final Examination

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1. Let F be a continuous, monotone increasing function on the real line. If A is an interval with endpoints a and b, let

$$\mu_F(A) := F(b) - F(a).$$

More generally, if A is a disjoint union of intervals

$$A = \bigcup_{i=1}^{N} A_i$$

let $\mu_F(A) = \sum_{i=1}^{N} \mu_F(A_i)$. Prove that μ_F is a measure on the ring $\mathcal{R}_{Leb}$; that is, prove it is countably additive.

2. Let $a > 0$ and $f(x)$ be a continuous function on $[-a, a]$. Compute the limit

$$\lim_{N\to\infty} N^{1/2} \int_{-a}^{a} e^{-Nx^2/2} f(x)dx.$$

Justify each step carefully. (Hint: Make a change of variables in the integral).

3. Let $(X, \mathcal{F}, \mu)$ be a measure space, A and B be measurable subsets of X and $S(A, B) := (A - B) \cup (B - A)$. Show that, if $\mu(S(A, B)) = 0$, then, for every nonnegative measurable function f,

$$\int_A f d\mu = \int_B f d\mu.$$

4. Let R_i be the i-th Rademacher function, and let

$$f_k(x) = \sum_{i=1}^{k} \left(\frac{1}{2^i}\right) R_i(x).$$

Compute

$$\int_{[0,1]} e^{tf_k(x)} dx.$$

(Hint: use independence).

5. Let $\chi_{[-1,1]}$ be the indicator function of the interval $[-1, 1]$ (ie. its characteristic function).
(i) Compute $\widehat{\chi_{[-1,1]}}(y)$.
(ii) Use the Plancherel formula to compute

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx.$$