1. A metric space \((X, \rho)\) is said to be \textit{locally compact} if for every \(x \in X\), there exists an open set \(U_x\), with \(x \in U_x\) and \(\overline{U_x}\) compact. Prove that (i) \(\mathbb{R}^n\) is locally compact, and (ii) \(\ell^1\) is not locally compact.

2. Let \((X, \rho)\) be a metric space with at least 2 distinct points. Show that there exists a non-constant continuous function \(X \to \mathbb{R}\). If further \(X\) is connected, show that \(X\) must be uncountable.

3. (a) Define what is meant by a connected component of a metric space \((X, \rho)\). If \(E \subset X\) is non-empty, open, closed and connected, show that \(E\) is a component.

(b) If \((X, \rho)\) is a connected metric space, \(f : X \to Y\) is a continuous map, show that \(f(X)\) is connected.

Show that the circle \(T = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}\) is not homeomorphic to the closed interval \([0, 1]\).

4. (a) State the Ascoli-Arzela theorem on the relative compactness of sets of continuous functions on a compact metric space.

(b) Let \(\alpha > 0\), \(M > 0\) be fixed; let
\[
E = \{f \in C([0, 1]) : |f(x)| \leq M, |f(x) - f(y)| \leq M|x - y|^{\alpha}, \forall x, y \in [0, 1]\}.
\]
Show that \(E\) is relatively compact in \(C[0, 1]\).

(c) Give an example of a countable infinite set of functions on a closed, bounded interval, which is neither equicontinuous nor uniformly bounded.

5. (a) State the Stone-Weierstrass theorem for real-valued functions on a compact metric space.

(b) For any bounded closed interval in \(\mathbb{R}\), let \(\mathcal{P}[a, b]\) denote the real vector space of real-valued polynomials defined on \([a, b]\).

If \(f : [a, b] \times [c, d] \to \mathbb{R}\) is continuous, show that \(f\) can be uniformly approximated by functions of the form
\[
g_1(x)h_1(y) + \cdots + g_k(x)h_k(y),
\]
\(k \in \mathbb{N}, g_j \in \mathcal{P}[a, b], h_j \in \mathcal{P}[c, d], j = 1, \cdots, k; x \in [a, b], y \in [c, d].\)
6. (a) State the inverse and implicit function theorems.

(b) Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be the map \( f(x, y) = (u, v) \) with \( u = x, \ v = xy \). Find \( f' \), and determine at what points \((x, y)\) the map \( f \) is locally one to one. Is the map one to one on all of \( \mathbb{R}^2 \)? Find the image under \( f \) of the rectangle \( \{(x, y) : 1 \leq x \leq 2, \ 0 \leq y \leq 2\} \).

(c) Under what conditions do the equations

\[
F(x, y, z) = 0, \quad G(x, y, z) = 0
\]

determine \( x, y \) as functions \( x = f(z), \ y = g(z) \) of \( z \), near a point \((x_0, y_0, z_0)\) which satisfies these two equations? Apply this to the functions

\[
F(x, y, z) = z^2 + xy - a, \quad G(x, y, z) = z^2 + x^2 - y^2 - b,
\]

where \( a, b \in \mathbb{R} \) and are constant. Compute \( f'(x) \) and \( g'(z) \) for these functions, and a point \((x_0, y_0, z_0)\) which satisfies the equations as well as the conditions you have found.

7. Prove or disprove each of the following: If the statement is true give a proof, if it is false, give a counter example.

(a) If \( F \) is closed, \( F \subseteq \mathbb{R}, \ F \) uncountable then the interior \( F^\circ \neq \emptyset \).

(b) If \( X \) is compact, \( f : X \to Y \) is continuous, then \( f(X) \) is compact.

(c) \( \{\sin nx : n \in \mathbb{N}\} \) is equicontinuous on \([0, 2\pi]\).
McGILL UNIVERSITY

FACULTY OF SCIENCE

SUPPLEMENTAL/DEFERRED EXAMINATION

MATHEMATICS 189-354A

ANALYSIS III

Examiner: Professor J.R. Choksi
Associate Examiner: Professor S.W. Drury
Date: Tuesday, May 4, 1999
Time: 9:00 A.M. - 12:00 Noon.

INSTRUCTIONS

NO CALCULATORS ARE PERMITTED.
All questions carry equal marks.
Attempt any 6 (SIX) questions.

This exam comprises the cover and 2 pages of questions.