1. If $A$ and $B$ are two subsets of $\mathbb{R}^n$, define

$$A + B = \{a + b : a \in A, b \in B\}.$$ 

Prove the following

(a) If $A$ is open $B$ arbitrary, then $A + B$ is open.
(b) If $A$ and $B$ are both compact, then $A + B$ is compact.
(c) If $A$ is compact, $B$ is closed, then $A + B$ is closed.

2. Let $(X, \rho)$ be a metric space, $\{f_n\}$ a sequence of continuous, real valued functions on $X$.

(a) If $f_n$ converges uniformly on $X$ to a function $f$, show that $f$ is continuous.
(b) If further $x_n \to x$ in $X$, show that $f_n(x_n) \to f(x)$.

3. (a) If $X$ is a connected metric space, and $f$ is a continuous function from $X$ to a metric space $Y$, show that $f(X)$ is connected.

(b) Let $f$ be a continuous function mapping the closed unit interval $[0, 1]$ into itself. Prove that $f(x) = x$ for at least one $x \in [0, 1]$.

4. Let $\{f_n\}$ be a sequence of differentiable, real-valued functions on $[0, 1]$, and suppose there exists $M > 0$ such that

$$|f_n(x)| \leq M, \quad |f_n'(x)| \leq M, \quad \forall n \in \mathbb{N}, \ x \in [0, 1].$$

Show that $\{f_n\}$ has a uniformly convergent subsequence.

5. State the Stone-Weierstrass Theorem for $C(\mathbb{R},X)$, $X$ a compact metric space.

Let $C_0$ be the (closed) subspace of $C([0,2\pi])$ consisting of continuous functions $f$ such that $f(0) = f(2\pi)$. Show that $C_0$ can be identified in a natural way with the space $C(\mathbb{T})$ where $\mathbb{T}$ is the unit circle centre the origin in $\mathbb{R}^2$ (i.e., the set

$$\{(x,y) : x^2 + y^2 = 1\}).$$

Hence show that if

$$\mathcal{T} = \left\{a_0 + \sum_{j=1}^n a_j \cos jt + b_j \sin jt : a_0, a_j, b_j \in \mathbb{R}, \ n \in \mathbb{N}, \ 0 \leq t \leq 2\pi\right\},$$

then $\mathcal{T}$ is uniformly dense in $C_0$, i.e. every function in $C_0$ is the uniform limit of a sequence of functions in $\mathcal{T}$. 
6. (a) State the inverse and implicit function theorems.

(b) Let \( f \) be a \( C^1 \) function \( \mathbb{R} \rightarrow \mathbb{R} \), and for \( (x,y) \in \mathbb{R}^2 \), let \( u = f(x), \ v = -y + xf(x) \).

If \( f'(x_0) \neq 0 \) for some point \( x_0 \in \mathbb{R} \), show that the map \( g(x,y) = (u,v) \) is invertible near \( (x_0,y) \) for all \( y \in \mathbb{R} \), and the inverse is given by \( x = f^{-1}(u), \ y = -v + uf^{-1}(u) \).

[State carefully what results you use, including results about functions of one variable.]

(c) Is it possible to solve the equations

\[
xy^2 + xzu + yv^2 = 3 \\
u^3yz + 2xv - u^2v^2 = 2
\]

for \( u = u(x,y,z), \ v = v(x,y,z) \) near \( (x,y,z) = (1,1,1), \ (u,v) = (1,1) \)? Compute \( \frac{\partial v}{\partial y} \) at \( (1,1,1) \).

7. Prove or disprove any 2 (two) of the following.

(a) A path connected metric space is connected.

(b) The space \( \ell^\infty \) is separable.

(c) In a metric space two disjoint closed sets are contained in disjoint open sets.
INSTRUCTIONS

NO CALCULATORS ARE PERMITTED.
All questions carry equal marks.
Attempt any 6 (SIX) questions.