1. Carefully state the following theorems.
   (i) (5 marks) The Heine–Borel Theorem.
   (ii) (5 marks) The Baire Category Theorem.
   (iii) (5 marks) The Tietze Extension Theorem.
   (iv) (5 marks) The Picard Existence Theorem.

2. (i) (5 marks) Write down the definition of the sequence space ℓ^∞ and its norm.
   (ii) (15 marks) Prove from first principles that ℓ^∞ is complete.

3. (20 marks) Let X be a compact metric space and let $(U_\alpha)_{\alpha\in I}$ be a arbitrary family of open subsets such that $X = \cup_{\alpha\in I} U_\alpha$. Show that there exists a strictly positive real number δ such that for each $x \in X$, there exists $\alpha \in I$ such that $U(x, \delta) \subseteq U_\alpha$. Here the notation $U(x, \delta)$ stands for $\{\xi; \xi \in X, d(x, \xi) < \delta\}$.

4. (i) (7 marks) Let A and B be disjoint closed subsets of a metric space X. Show that there necessarily exist disjoint open subsets U and V such that $A \subseteq U$ and $B \subseteq V$.
   (ii) (6 marks) Construct explicitly disjoint closed subsets A and B of $\mathbb{R}$ such that $\inf\{|a - b|; a \in A, b \in B\} = 0$.
   (iii) (7 marks) Suppose that A and B are disjoint closed subsets of $\mathbb{R}^d$ and in addition that B is bounded. Show that $\inf\{\|a - b\|; a \in A, b \in B\} > 0$. Here $\| \ \|$ denotes the Euclidean norm on $\mathbb{R}^d$.

5. (i) (10 marks) Let Ω be a connected open subset of $\mathbb{R}^d$. Prove that any two points of Ω can be joined by a piecewise linear path lying entirely in Ω.
   (ii) (10 marks) Let Ω be a connected open subset of $\mathbb{R}^d$ and let $f : \Omega \longrightarrow \mathbb{R}$ be a differentiable function with everywhere vanishing total derivative. Show that f is constant on Ω.

6. (i) (5 marks) State carefully the Implicit Function Theorem.
   (ii) (5 marks) State carefully the Parametrization Theorem.
   (iii) (10 marks) Show that the set of points $(x, y, z) \in \mathbb{R}^3$ satisfying the equation $(x + y + z)^3 + 396 + 20xyz = 0$ is an infinitely differentiable surface.

7. (i) (5 marks) What is meant by the *Hessian* of a real-valued function.

(ii) (5 marks) State and prove a theorem relating to the symmetry of the Hessian.

(iii) (5 marks) State a theorem relating the Hessian to local minimum points.

(iv) (5 marks) Show that the function $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$\varphi(x, y) = (x - y^2)(x - 3y^2)$$

does not have a strict local minimum at the origin $(x, y) = (0, 0)$. Show that nevertheless the restriction of φ to every line through the origin does have a strict local minimum at the origin.

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