Final Examination
Mathematics 189-346B
Number Theory

Justify all your assertions

Part I

1. (a) Let a, b, m, n be positive integers with a, b relatively prime and $m^a = n^b$. Show that there is a positive integer c such that $m = c^b$, $n = c^a$.

   (b) Show that $1 + 1/2 + \ldots + 1/n$ is not an integer for $n > 1$.

2. (a) Give a method for computing $a^b \pmod{c}$ by taking products of certain successive squares of a modulo c. Use this method to compute $2^{45} \pmod{91}$.

   (b) Given that $2^{693} \equiv 512 \pmod{1387}$, what can you say about the primality of 1387.

3. (a) If c is an integer relatively prime to n such that $c^m \equiv 1 \pmod{n}$ for some positive integer m with $c^{m/p} \not\equiv 1 \pmod{n}$ for each prime divisor p of m, show that m is the order of c modulo n.

   (b) Show that 2 is a primitive root modulo 49. Is it a primitive root modulo 343?

4. (a) Given that 2 is a primitive root modulo 49, find all solutions of $x^5 \equiv 2 \pmod{49}$.

   (b) Find all solutions of $x^3 + 2x - 3 \equiv 0 \pmod{49}$.

Part II

5. (a) Find all primes p such that 10 is a square modulo p.

   (b) Determine whether or not 137 is a square modulo 401.

6. Using the fact that $4001x^2 + 6204xy + 2405y^2$ is a quadratic form with discriminant -4, find a representation of 4001 as a sum of two squares.

7. Find all integer solutions of the system

$$\begin{aligned} x + 2y + 4z &= 3 \\ 2x + 7y - z &= -6. \end{aligned}$$

8. Using the fact that the Euler function φ is multiplicative, show that

$$\sum_{d \mid n} \varphi(d) = n.$$

   Using the Möbius inversion formula, show how to deduce a formula for $\varphi(n)$.