1. Solve the following differential equations and determine that solution satisfying the initial condition if one is given:

(a) \( x y' - 3 y = x^4 \), \( y(1) = 3 \);
(b) \( x^2 y' = y^2 - 2x^2 \), \( y(1) = 0 \);
(c) \( y' = (x + y - 2)^2 \).

2. Consider the nonhomogeneous second-order differential equation

\[(1 - x) y'' + x y' - y = 4 (x - 1)^2 e^{-x} .\]

(a) Given that \( y_1 = e^x \) is a solution of the homogeneous equation, use the method of reduction of order to find that \( y_2 = x \) is a second homogeneous solution.

(b) Now, obtain the general solution of the original nonhomogeneous ODE.  

[Useful integral: \( \int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1) \).]

3. Find the general solution in terms of real valued functions of \( x \) for

\[ y^{iv} + 3 y'' - 4 y = 5 e^x .\]

4. Find a solution of the nonlinear ODE

\[ 2 y y'' = 1 + (y')^2 \]

satisfying the initial conditions \( y(0) = y'(0) = 1 \).

5. Determine all real eigenvalues and eigenfunctions for the boundary-value problem

\[ x^2 y'' + x y' + 4 \lambda^2 y = 0, \quad y(1) = 0 \text{ and } y'(e^\pi) = 0 .\]
6. Consider the problem of finding series solutions in powers of $x$ for the equation

$$2x y'' + (3 - x) y' + (\gamma - 1) y = 0,$$

where $\gamma$ is a constant.

(a) Determine the indicial equation and its roots;

(b) Find the recurrence relation for successive terms in the series;

(c) Show that polynomial solutions exist when $\gamma$ is an integer and write down such a solution for the case $\gamma = 3$;

(d) A second linearly independent solution $y_2(x)$, say, is in the form of an infinite series. Write down the general solution for the case $\gamma = 3$ including at least the first two terms of $y_2(x)$.

7. (a) Solve the integral equation $y(t) - 2 \int_0^t e^{t-\tau} y(\tau) d\tau = e^{2t}$ by taking the Laplace transform of both sides.

(b) Use the Laplace transform to solve the initial-value problem

$$y'' - 2y' + 5y = \delta(t - \pi) \cos t, \quad y(0) = y'(0) = 1.$$