1. (28 points) Solve each of the following:

(a) $y' = \dfrac{4y - 3x}{2x - y}$;

(b) $yy' + y^2 + x + x^2 = 0$;

(c) $y' = (x + y + 1)^2 - 2$; (Hint: introduce z = x+y+1)

(d) $y'' + 2x[y']^2 = 0,\ y(0) = 0,\ y'(0) = 1.$

For the equations (a) and (b) determine for which data (x_0, y_0) the existence of a solution with the initial value condition $y(x_0) = y_0$ is assured. Give reasons.

2. (10 points) Solve the equation $t^2y' + 2ty - y^3 = 0$ subject to the initial condition $y(1) = -1$. What is the maximum interval of existence for that solution?

3. (10 points) Find the general solution of $y'' + y' - 6y = te^{-t}$. For which initial conditions $y(0) = y_0$ and $y'(0) = y_1$, does the corresponding solution satisfy $lim_{t\to\infty} y(t) = 0$

4 (10 points) Find a formula for the solution of $t^2y'' - t(t + 2)y' + (t + 2)y = f(t)$ subject to the initial conditions $y(1) = y'(1) = 0$ given that $y_1(t) = t$ is a solution of the corresponding homogeneous equation.

5. (17 points) Use Laplace transforms

(a) to solve the initial value problem

$$y'' + 4y = u_\pi(t) - u_{2\pi}(t) + \delta(t - 1), \quad y(0) = a,\ y'(0) = b.$$

(b) to find a convolution formula for the solution of

$$y'' + 4y = f(t), \quad y(0) = 0,\ y'(0) = 0.$$

Then set $f(t) = u_\pi(t) - u_{2\pi}(t) + \delta(t - 1)$ in the formula you have found in (b) and verify your answer by comparison with your solution in (a).

6 (25 points) Consider Chebyshev's equation $(1 - x^2)y'' - xy' + \alpha^2 y = 0$.

(a) Find the general solution in an interval about $x = 0$ using power series expansions about that point.

(b) What can you say about the radius of convergence of the solutions you have found in (a). Why?

(c) Show that if α is an integer n then Chebyshev's equation has a polynomial solution of degree n; suitably normalized these are denoted by $T_n(x)$.

(d) Show that for $n \neq m$

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} T_n(x) T_m(x)\, dx = 0;$$

(e) Show that $x = +1$ and $x = -1$ are regular singular points for the equation and determine the corresponding indicial polynomials.

(f) Without calculating the coefficients. but using information obtained from the indicial equation, write down the **form** of the general solution expressed in terms of power series centered at $x = 1$.