1. (a) Determine the domain of analyticity of \( f(z) = \log i(z - 1) \).
   (b) Make suitable branch cuts and define a branch \( f(z) \) of \((z^2 + 1)^{1/2}\) that is defined on the real axis and such that \( f(0) = -1 \). Find \( f(-1) \). Justify your answer.

2. Let \( f(z) \) be a complex function and \( z_0 \in \mathbb{C} \).
   (a) Define the following concepts:
      i. \( z_0 \) is an isolated singularity of \( f \).
      ii. \( z_0 \) is a removable singularity of \( f \).
      iii. \( z_0 \) is a pole of \( f \).
      iv. \( z_0 \) is an essential isolated singularity of \( f \).
   (b) Explain how to extend the definitions in (a) to the case \( z_0 = \infty \).

3. For the following functions \( f(z) \) determine the type of singularity at the point \( z_0 \) indicated.
   (a) \( f(z) = e^{-1/z} \sin z^2; \ z_0 = 0 \).
   (b) \( f(z) = \frac{1 + z}{1 - z}; \ z_0 = \infty \).
   (c) \( f(z) = \log \frac{z - 1}{z + 1}; \ z_0 = 1 \).
   (d) \( f(z) = \frac{1}{1 - \cos z}; \ z_0 = 0 \).

4. For the functions \( f(z) \) and points \( z_0 \) of 3), determine whether \( f(z) \) has a Laurent expansion at \( z_0 \). Where possible, find the order and the residue of \( f \) at \( z_0 \).

5. (a) Suppose \( f(z) \) has an isolated singularity at \( z_0 \in \mathbb{C} \). Show that the following are the same:
      i. the coefficient of \((z - z_0)^{-1}\) in the Laurent expansion of \( f(z) \) at \( z_0 \).
      ii. \( \frac{1}{2\pi i} \int_{|z-z_0|=\varepsilon} f(z) \, dz \) (for \( 0 < \varepsilon \) sufficiently small).
   (b) Suppose that \( f(z) \) has a pole or removable singularity at \( z_0 \). Let \( g(z) = \frac{f'(z)}{f(z)} \). Show that \( \text{Res}(g(z); z_0) \) is defined and equals the order of \( f(z) \) at \( z_0 \).

6. Determine the following integrals. Use residues and contour integration where appropriate. Justify your steps.
   (a) \( \int_{|z|=2} \frac{1}{z^2 + z + 1} \, dz \);
   (b) \( \int_0^{2\pi} \frac{\sin \theta}{2 + \cos \theta} \, d\theta \);
   (c) \( \int_0^\infty \frac{\cos x}{x^2 + 1} \, dx \).
McGILL UNIVERSITY
FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-316A

FUNCTIONS OF A COMPLEX VARIABLE

Examiner: Professor K.P. Russell
Associate Examiner: Professor J.C. Taylor

Date: Thursday, December 10, 1998
Time: 2:00 P.M. - 5:00 P.M.

This exam comprises the cover and 1 page of questions.