McGILL UNIVERSITY
FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 315
ORDINARY DIFFERENTIAL EQUATIONS

Examiner: Professor J.J Xu
Associate Examiner: Professor J. Labute

Date: Friday April 13, 2007
Time: 2:00PM - 5:00PM

INSTRUCTIONS

This is a closed book exam
Answer all questions in the exam booklets provided.
Faculty standard calculators are permitted.
A table of Laplace Transforms has been provided.

This exam comprises the cover, 3 pages of 8 questions and 1 page of tables.
1. (10pts) Given the equation

\[ 2y(y + 2x^2) dx + x(4y + 3x^2) dy = 0. \]

(a) Show that this is not an exact equation,

(b) Determine the values of the constants \( \alpha \) and \( \beta \), such that \( \mu(x, y) = x^\alpha y^\beta \) is an integrating factor for this equation;

(c) By using the integral factor found above, derive the general solution of the equation.

2. (5pts) Perform the phase line analysis for the following autonomous equation:

\[ \frac{dy}{dt} = y(y - 1)^2(y - 3), \]

and determine that

- its equilibrium states;
- the type of each equilibrium state,
- the stability property of each equilibrium state,
- sketch the integral curves in the physical plane \((t, y)\), based on the above phase line analysis without solving the equation.

3. (15pts) Find the general solution for the following equations:

(a) \((D^2 - 2D + 2)^2(D^2 - 1)y = 0;\)

(b) \((D^2 + 4)y = 16x \cos 2x;\)

4. (15pts)

(a) Find all values of \( \alpha \) for which all solutions of

\[ x^2 y'' + \alpha xy' + \frac{5}{2} y = 0 \]

approach to zero as \( x \to \infty. \)
(b) Find the general solution for the following equations by using the method of variation of parameters:

\[ x^2y'' - 4xy' + 6y = x^4 \sin x, \quad (x > 0). \]

5. (15pts)

Given the following equation

\[ x^2y'' + \frac{1}{2}(x + \sin x)y' + y = 0, \]

(a) Find all the regular singular points

(b) Derive the indicial equation and the exponents at the singularity for each regular singular point;

(c) Determine whether the given equation has a solution that is bounded near the regular singular point, has all solutions bounded near the regular singular point, or has no non-zero solution bounded near the regular singular point.

6. (10pts) Given the equation

\[ 2xy'' + y' + xy = 0, \]

(a) Show that \( x = 0 \) is a regular singular point of the given equation and give the roots of the indicial equation;

(b) Determine the recurrence formula for the coefficients in the Frobenius series expansion of the solution near \( x = 0 \);

(c) Find at least the first four terms of two linear independent solutions: \( y_1(x), y_2(x) \).
7. (10pts) (Choose one from two problems. You may get bonus points, if you solved two.) Find the Laplace transform of the following functions:

(a) \[ f(t) = 4 \cos^2 bt, \quad (b \text{ constant}); \]

(b) \[
   f(t) = \begin{cases} 
   0, & 0 \leq t \leq 1 \\
   t, & 1 < t \leq 2 \\
   0, & t > 2. 
   \end{cases}
\]

8. (10pts) (Choose one from two problems. You may get bonus points, if you solved two.) Find the inverse Laplace transform of the following functions:

(a) \[
   F(s) = \frac{2s + 3}{(s - 2)(s^2 + 1)},
\]

(b) \[
   F(s) = \frac{e^2e^{-4s}}{2s - 1}.
\]

9. (10pts) Solve the following IVP's with the Laplace transform method:

\[
y'' + 4y = \sin t - u_{2\pi}(t) \sin(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0.
\]
<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$Lf(s) = \int_0^\infty f(t)e^{-st}dt$</th>
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<th>$Lf(s) = \int_0^\infty f(t)e^{-st}dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^n, n \in \mathbb{N}$ [ \frac{n!}{s^{n+1}}, \Re s &gt; 0 ]</td>
<td>$u_a(t), a &gt; 0$ [ \frac{e^{-as}}{s}, \Re s &gt; 0 ]</td>
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<tr>
<td>$1$ [ s^{-1}, \Re s &gt; 0 ]</td>
<td>$u_a(t)g(t-a), a &gt; 0$ [ e^{-as}Lg(s) ]</td>
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<tr>
<td>$t$ [ s^{-2}, \Re s &gt; 0 ]</td>
<td>$e^{at}g(t)$ [ Lg(s-a) ]</td>
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<tr>
<td>$t^\nu, \nu &gt; -1$ [ \frac{\Gamma(\nu+1)}{s^{\nu+1}}, \Re s &gt; 0 ]</td>
<td>$g(ct), c &gt; 0$ [ \frac{1}{c}Lg\left(\frac{s}{c}\right) ]</td>
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<tr>
<td>$e^{at}$ [ (s-a)^{-1}, \Re s &gt; a ]</td>
<td>$\delta_a(t) = \delta(t-a), a &gt; 0$ [ e^{-as}, \Re s &gt; 0 ]</td>
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<tr>
<td>$\sin(\omega t)$ [ \frac{\omega}{s^2 + \omega^2}, \Re s &gt; 0 ]</td>
<td>$f * g(t) = \int_0^t f(t-u)g(u)du$ [ Lf(s)Lg(s) ]</td>
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<tr>
<td>$\cos(\omega t)$ [ \frac{s}{s^2 + \omega^2}, \Re s &gt; 0 ]</td>
<td>$f^{(n)}(t), n \in \mathbb{N}$ [ s^nLf(s) - \sum_{k=0}^{n-1} s^{n-k-1}f^{(k)}(0) ]</td>
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<tr>
<td>$e^{at}\sin(\omega t)$ [ \frac{\omega}{(s-a)^2 + \omega^2}, \Re s &gt; a ]</td>
<td>[ \frac{df}{dt}(t) ] [ sLf(s) - f(0) ]</td>
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<tr>
<td>$e^{at}\cos(\omega t)$ [ \frac{s-a}{(s-a)^2 + \omega^2}, \Re s &gt; a ]</td>
<td>[ \frac{d^2f}{dt^2}(t) ] [ s^2Lf(s) - sf(0) - f'(0) ]</td>
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<tr>
<td>$t^n e^{at}, n \in \mathbb{N}$ [ \frac{n!}{(s-a)^{n+1}}, \Re s &gt; a ]</td>
<td>$t^n f(t), n \in \mathbb{N}$ [ (-1)^n \frac{d^nLf}{ds^n}(s) ]</td>
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