McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 315

ORDINARY DIFFERENTIAL EQUATIONS

Examiner: Professor W. Jonsson
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Date: Thursday April 27, 2006
Time: 2:00PM - 5:00PM

INSTRUCTIONS

1. Please answer all questions in the exam booklets provided.
2. Questions are not necessarily of equal weight.
3. This is a closed book exam.
4. Calculators are neither needed nor allowed.
5. Use of a regular dictionary is not permitted.
6. A table of Laplace Transforms is supplied.
7. This exam comprises the cover page, and 1 page of 6 questions.

Ordinary Differential Equations 315
Final Examination April 2006

1. (a) Solve the initial value problem

$$y' + 4y = xy^3 \text{ with } y(0) = 1$$

(b) Solve the following initial value problem

$$(2x^2 + y)dx + (x^2y - x)dy = 0 \text{ with } y(1) = 2$$

(a) Solve the following homogeneous linear differential equation

$$x^2y'' + xy' + y = 0 \text{ with } y(1) = 0, y'(1) = 1$$

(b) Using variation of parameters, solve

$$2y'' - 3y' + y = e^x \text{ with } y(0) = 1,\ y'(0) = 0$$

In this question the majority of the marks are for carrying out the method indicated.

2. Solve the following differential equation

$$(1 - x^2)y'' - 2xy' + 12y = 0$$

in the form $= \sum_{n=0}^{\infty} a_n x^n$ by

(a) finding the recurrence relation on the coefficients and show that there is a polynomial solution.
(b) displaying a non-trivial polynomial solution.
(c) finding a formula for the coefficients as a function of n for a solution which is not a polynomial.

3. A non-trivial solution of the ordinary differential equation

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$$

is to be found using the method of Frobenius (i.e. a solution of the form $y_1 = x^\alpha \sum_{n=0}^{\infty} a_n x^n$ with $a_0 \neq 0$) by

(a) first finding the indicial equation and its roots.
(b) then finding the recurrence relation for the coefficients.
(c) solving the recurrence relation for the coefficients in terms of a_0 and a function of n using the larger root of the indicial equation.

4. (a) Find the Laplace transform $L\{f(t)\}$ where

$$f(t) = \int_0^t \tau \cos(2\tau) e^{4(t-\tau)} d\tau.$$

(b) Find the inverse Laplace transform $L^{-1}\{F(s)\}$ where

$$F(s) = \frac{(s+2)e^{-2s}}{s(s^2 + 2s + 1)}.$$

5. Using Laplace transform methods, solve

$$y'' - y' = g(t) + \delta(t-3)e^t \text{ with } y(0) = 0, y'(0) = 0,$$

, where the function $g(t)$ is defined by

$$g(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 2 & \text{otherwise} \end{cases} .$$

6. Apply the improved Euler method to the differential equation

$$y' = y + 2t \text{ with } y(1) = 2$$

using step size $h = 0.1$ to compute an approximation to $y(1.2)$.

Show your work and display a table with the results of intermediate calculations.

LAPLACE TRANSFORM TABLE

$f(t)$	$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}dt$	$f(t)$	$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}dt$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}},\ \Re s > 0$	$u_a(t),\ a > 0$	$\frac{e^{-as}}{s},\ \Re s > 0$
1	$s^{-1},\ \Re s > 0$	$u_a(t)g(t-a),\ a > 0$	$e^{-as}\mathcal{L}g(s)$
t	$s^{-2},\ \Re s > 0$	$e^{at}g(t)$	$\mathcal{L}g(s-a)$
$t^\nu,\ \nu > -1$	$\frac{\Gamma(\nu+1)}{s^{\nu+1}},\ \Re s > 0$	$g(ct),\ c > 0$	$\frac{1}{c}\mathcal{L}g\left(\frac{s}{c}\right)$
e^{at}	$(s-a)^{-1},\ \Re s > a$	$\delta_a(t) = \delta(t-a),\ a > 0$	$e^{-as},\ \Re s > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2},\ \Re s > 0$	$f * g(t) = \int_0^t f(t-u)g(u)du$	$\mathcal{L}f(s)\mathcal{L}g(s)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2},\ \Re s > 0$	$f^{(n)}(t),\ n \in \mathbb{N}$	$s^n\mathcal{L}f(s) - \sum_{k=0}^{n-1} s^{n-k-1}f^{(k)}(0)$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2},\ \Re s > a$	$\frac{df}{dt}(t)$	$s\mathcal{L}f(s) - f(0)$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2},\ \Re s > a$	$\frac{d^2f}{dt^2}(t)$	$s^2\mathcal{L}f(s) - sf(0) - f'(0)$
$t^n e^{at},\ n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}},\ \Re s > a$	$t^n f(t),\ n \in \mathbb{N}$	$(-1)^n \frac{d^n\mathcal{L}f}{ds^n}(s)$