1. Given the equation

$$y' = (y-1)(y-2)^2(y-3) \quad (t > 0)$$

   (a) Sketch the phase plane diagram in the (y, y') plane.
   (b) What is the nature of its equilibria (stable, unstable or semi-stable)?
   (c) Predict the asymptotic behaviour as $t \to \infty$ of the solution satisfying $y(0) = 2.1$.

2. Solve the following equations:

   (a) $(3x^2 + y)dx + (x^2y - x)dy = 0$.
   (b) $(y^2 + 2xy)dx - x^2dy = 0$.

3. Given the following free vibration problem:

$$y'' + by' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 0$$

   for $b = 5,\ 4,\ 2$.

   (a) Find the solutions for each value of b.
   (b) Sketch the solutions.

4. Solve the following equations.

   (a) $x^2y'' + xy' - 2y = 0 \quad (x > 0)$.
   (b) $x^2y'' + 3xy' + 5y = 0 \quad (x > 0)$.
   (c) $(x-2)^2y'' - 7(x-2)y' + 7y = 0 \quad (x > 2)$.

5. Find the general solution for each of the following equations.

   (a) $y'' - 4y' + 5y = e^{5x} + x\sin 3x - \cos 3x$.
   (b) $xy'' - 3y' - \dfrac{3}{x}y = x^2$.

6. Given the following equations:

(a) $9x^2y'' + 9x^2y' + 2y = 0$.

(b) $(x^2 - 4)y'' + (x + 2)y' + 3y = 0$.

(c) $(4x \sin x)y'' - 3y = 0$.

(i) Classify each singular point as regular, or regular singular, or irregular singular.

(ii) Find the indicial equation and the exponents at the regular singular points.

(iii) Find at least the first three nonzero terms in the series expansion about $x = 0$ for the general solution of equation (a).

7. In terms of the Laplace transforms method, solve the following problems:

(i)
$$\begin{aligned} y'' - y' - 2y &= -8\cos t - 2\sin t \\ y\left(\frac{\pi}{2}\right) &= 1 \\ y'\left(\frac{\pi}{2}\right) &= 0. \end{aligned}$$

(ii)
$$\begin{aligned} y'' + y &= g(t) \\ y(0) &= 0 \\ y'(0) &= 1 \\ g(t) &= \begin{cases} 1 \text{ if} & 0 < t \le \frac{\pi}{2} \\ 0 \text{ if} & t > \frac{\pi}{2} \end{cases} \end{aligned}$$

McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS 189-315B

ORDINARY DIFFERENTIAL EQUATIONS

Examiner: Professor J.J. Xu | Date: Thursday, April 13, 2000
Associate Examiner: Professor S.A. Maslowe | Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Calculators are permitted.

This exam comprises the cover, two pages of questions and one page of Laplace Transforms.