McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 315

ORDINARY DIFFERENTIAL EQUATIONS

Examiner: Professor J.J Xu — Date: Thursday December 8,2005.

Associate Examiner: Lahcen Laayouni — Time: 9:00 am - 12:00pm

INSTRUCTIONS

Answer all questions in the exam booklets provided.
This is a closed book exam
Faculty standard calculators are permitted.
A table of Laplace Transforms has been provided.
DICTIONARIES ARE ALLOWED
EXAM IS PRINTED DOUBLE-SIDED

This exam comprises the cover, 2 pages of 7 questions and 1 page of tables.

1. **(10 pts)** Find the general solutions for each of the following equations in its definition interval:

   (i)
$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = \sqrt{9x^2 + y^2};$$

   (ii)
$$x^2 y \mathrm{d}x + y(x^3 + \mathrm{e}^{-3y}\sin y)\mathrm{d}y = 0;$$

2. **(5 pts)** Given a family of curves $\{C\}$ with the function
$$y = \frac{1}{x^2 + c}.$$
   Find the equation of orthogonal trajectories to $\{C\}$.

3. **(15 pts)** Find the general solutions for each of the following equations:

   (a)
$$(D-1)(D^2+4)^2 y = x\mathrm{e}^{2x} - 2\sin 2x;$$

   (b)
$$x^2 y'' + 4xy' + 2y = \cos x \quad (x > 0),$$
   (use the method of variation of parameters).

4. **(15pts)** Given the following equation
$$(x-2)^2 y'' + (x-2)\mathrm{e}^x y' + \frac{4}{x} y = 0,$$

   (a) Classify each singular point of the given equation as regular or irregular;

   (b) Derive the indicial equation for each regular singular point;

(c) For each regular singular point, determine which of the following occurs without solving the equation:

(i) all solutions are bounded near the point;

(ii) there is a non-zero solution, but not all solutions are. bounded near the point ;

(iii) there is no non-zero solution bounded near the point.

5. **(20pts)** Given the equation

$$x^2y'' + xy' - (2 + x)y = 0,$$

(a) Show that $x = 0$ is a regular singular point of the given equation;

(b) Determine the recurrence formula for the coefficients in the Frobenius series expansion of the solution near $x = 0$;

(c) Find the first four terms of two linearly independent solutions: $\{y_1, y_2\}$ on the interval $(0, \infty)$.

6. **(20pts)**

(a) Find the Laplace transform of the following function:

$$f(t) = 2e^{3t} \sin t + 4e^{-t} \cos 3t;$$

(b) Sketch the given function; express it in terms of the unit step function then determine its Laplace transform.

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < 3, \\ e^{t-3}, & t \ge 3. \end{cases}$$

(c) Find the inverse Laplace transforms of the following functions:

$$\text{(i)} \quad F(s) = \frac{2s + 3}{s(s^2 - 2s + 5)}, \qquad \text{(ii)} \quad F(s) = \frac{e^{-s}(s + 6)}{s^2 + 9}.$$

7. **(15pts)** Solve the following IVP's with the Laplace transform method:

$$y'' + 3y' + 2y = 10u_{\pi/4}(t) \sin\left(t - \frac{4}{\pi}\right), \quad y(0) = 1, \ y'(0) = 0;$$

Laplace Transform Table

$f(t)$	$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}dt$	$f(t)$	$\mathcal{L}f(s) = \int_0^\infty f(t)e^{-st}dt$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}},\ \Re s > 0$	$u_a(t),\ a > 0$	$\frac{e^{-as}}{s},\ \Re s > 0$
1	$s^{-1},\ \Re s > 0$	$u_a(t)g(t-a),\ a > 0$	$e^{-as}\mathcal{L}g(s)$
t	$s^{-2},\ \Re s > 0$	$e^{at}g(t)$	$\mathcal{L}g(s-a)$
$t^\nu,\ \nu > -1$	$\frac{\Gamma(\nu+1)}{s^{\nu+1}},\ \Re s > 0$	$g(ct),\ c > 0$	$\frac{1}{c}\mathcal{L}g\left(\frac{s}{c}\right)$
e^{at}	$(s-a)^{-1},\ \Re s > a$	$\delta_a(t) = \delta(t-a),\ a > 0$	$e^{-as},\ \Re s > 0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2},\ \Re s > 0$	$f * g(t) = \int_0^t f(t-u)g(u)du$	$\mathcal{L}f(s)\mathcal{L}g(s)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2},\ \Re s > 0$	$f^{(n)}(t),\ n \in \mathbb{N}$	$s^n\mathcal{L}f(s) - \sum_{k=0}^{n-1} s^{n-k-1}f^{(k)}(0)$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2},\ \Re s > a$	$\frac{df}{dt}(t)$	$s\mathcal{L}f(s) - f(0)$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2},\ \Re s > a$	$\frac{d^2f}{dt^2}(t)$	$s^2\mathcal{L}f(s) - sf(0) - f'(0)$
$t^n e^{at},\ n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}},\ \Re s > a$	$t^n f(t),\ n \in \mathbb{N}$	$(-1)^n\frac{d^n\mathcal{L}f}{ds^n}(s)$