1. (11 marks) Solve the vibrating string problem:

\[
\begin{align*}
\psi_{xx} &= \psi_t; & 0 < x < 1, \ t > 0 \\
\psi(0, t) &= \sin t, \quad \psi(1, t) = 0 \\
\psi(x, 0) &= 0, \quad \psi_t(x, 0) = 1 - x
\end{align*}
\]

Leave your answer in simplest form.

2. (a) (5 marks) Determine the steady-state temperature distribution inside a sphere of radius “a” if the boundary temperature is maintained at a temperature of \(T_0(1 + 2\sin^2 \varphi)\).

Leave your answer in simplest form. You may assume that the general solution of Laplace’s equation in spherical coordinates with no \(\theta\) dependence, and converging for \(0 \leq \varphi \leq \pi\), is given by

\[
\psi(r, \varphi) = \sum_{n=0}^{\infty} \left[ A_n r^n + \frac{B_n}{r^{n+1}} \right] P_n(\cos \varphi).
\]

(b) (7 marks) Solve and interpret physically

\[
\nabla^2 \psi(r, \varphi) = 0, \quad r > a, \ 0 \leq \varphi \leq \pi.
\]

(i) \(\lim_{r \to \infty} [\psi(r, \varphi) - V_0 r \cos \varphi] = 0\).

(ii) \(\left. \frac{\partial \psi}{\partial r} \right|_{r=a} = 0\).

3. (11 marks) A sphere of radius \(b\) has its surface maintained at a temperature \(\beta\). There is a constant heat generation at the rate \(Q\). The initial temperature is \(f(r)\). Find the temperature at any point inside the sphere after time \(t\).

Leave your answer in simplest form.

Hints: (a) \(\psi = \psi(r, t)\); \quad (ii) \(\frac{1}{\alpha^2} \frac{\partial \psi}{\partial t} - \nabla^2 \psi = \frac{Q}{K}\).

4. (10 marks) Find the steady-state temperature distribution in the region below:
5. Solve and interpret physically:

(a) (12 marks) \( \nabla^2 \psi(r, z) = 0; \quad 0 < r < b, \quad 0 < z < \pi. \)

(i) \( \psi(r, 0) = 0, \quad \) (ii) \( \psi(r, \pi) = f(r), \quad \) (iii) \( \psi(b, z) = g(z). \)

Hint: Divide the problem into two parts.

(b) (9 marks) \( \nabla^2 \psi(r, z) = -F(r, z); \quad 0 < r < b, \quad 0 < z < \pi. \)

(i) \( \psi(r, 0) = 0, \quad \) (ii) \( \psi(r, \pi) = f(r), \quad \) (iii) \( \psi(b, z) = g(z). \)

6. (8 marks) A linear operator \( \mathcal{T} \) maps the vector \( 5\hat{i} - 3\hat{j} \) onto \( 24\hat{i} + \hat{j} \) and the vector \(-3\hat{i} + 7\hat{j} \) onto \(-4\hat{i} + 15\hat{j} \). Through what angle must the \( \hat{i}, \hat{j} \) basis vectors be rotated in order for the matrix representative of \( \mathcal{T} \) to be diagonal? It is sufficient to give the cosine of the required angle.

7. (a) (8 marks) The moment of inertia matrix is given by

\[
\begin{bmatrix}
3 & 2 & 1 \\
2 & 3 & 1 \\
1 & 1 & 4 \\
\end{bmatrix}
\]

when referred to the \( xyz \) coordinate system. Find the principal moments of inertia and unit vectors along the three principal axes.

(b) (3 marks) Use matrix methods to identify and sketch

\[3x_1^2 + 3x_2^2 + 4x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3 = 36.\]

(c) (3 marks) Obtain the matrix of transformation from the \( (x_1, x_2, x_3) \) axes to those axes with respect to which the equation has no cross terms making certain that this transformation is a rotation. Explain.

(d) (4 marks) Does

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-[3x^2+3y^2+4z^2+4xy+2xz+2yz]} dxdydz
\]

converge? If so, evaluate the integral.

8. (a) (6 marks) Solve the system

\[
\begin{align*}
\dot{x}_1 &= 5x_1 + 4x_2; \quad x_1(0) = 2 \\
\dot{x}_2 &= -x_1 + x_2; \quad x_2(0) = -3
\end{align*}
\]

by using the exponential matrix method.

(b) (6 marks) Write down the Green’s matrix for the above system and use it to solve

\[
\begin{align*}
\dot{x}_1 &= 5x_1 + 4x_2 + 2; \quad x_1(0) = 2 \\
\dot{x}_2 &= -x_1 + x_2 - 1; \quad x_2(0) = -3.
\end{align*}
\]
McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS 189-266B

LINEAR ALGEBRA & BOUNDARY VALUE PROBLEMS

Examiner: Professor C. Roth  
Associate Examiner: Professor N.G.F. Sancho  
Date: Wednesday, April 28, 1999  
Time: 9:00 A.M. - 1:00 P.M.

INSTRUCTIONS

Faculty Standard Calculators are permitted.

This exam comprises the cover, 2 pages of questions and 1 page of useful information.