MARKS

(6) 1. (a) Find the eigenvalues and eigenfunctions of

$$x^2y'' + 3xy' = \lambda y;\ 1 \le x \le e$$
$$y(1) = 0,\ y(e) = 0$$

(5) (b) Expand $f(x)$, piecewise smooth, in terms of the eigenfunctions in (a), leaving your answer in simplest form.

(2) 2. (a) Verify the orthogonality of the Legendre polynomials of odd order for the interval $[0, 1]$

$$\int_0^1 P_n(x)P_m(x)dx = \frac{\delta nm}{2n+1}\ .$$

(8) (b) Find the potential $\psi(r, \varphi)$ in the infinite region $r > b,\ 0 < \varphi < \frac{\pi}{2}$, if $\psi = 0$ on the plane portion of the boundary ($\varphi = \frac{\pi}{2},\ r > b$), $\psi \to 0$ as $r \to \infty$ and $\psi = f(\cos\varphi)$ on the hemispherical portion of the boundary ($r = b,\ 0 \le \varphi < \frac{\pi}{2}$).

(5) (c) Consider also the special case $f(\cos\varphi) = \cos^3\varphi$. Leave your answer in simplest form.

Hints: (a) You may assume that the general solution of Laplace's equation in spherical coordinates with no θ dependence, and converging for $0 \le \varphi \le \pi$, is given by

$$\psi(r, \varphi) = \sum_{n=0}^{\infty} \left[A_n r^n + \frac{B_n}{r^{n+1}}\right] P_n(\cos\varphi).$$

(b) Show that only Legendre polynomials of odd order are needed here.

(12) 3. A sphere of radius b has its surface maintained at a temperature β. There is a constant heat generation at the rate Q. The initial temperature is $f(r)$. Find the temperature at any point inside the sphere after time t.
Leave your answer in simplest form.

Hints: (a) $\psi = \psi(r,t)$; (b) $\dfrac{1}{\alpha^2}\dfrac{\partial \psi}{\partial t} - \nabla^2 \psi = \dfrac{Q}{K}$.

4. Solve and interpret physically:

(13) (a) $\nabla^2 \psi(r,z) = 0; \quad 0 < r < b,\ 0 < z < \pi.$

(i) $\psi(r,0) = 0$, (ii) $\psi(r,\pi) = f(r)$, (iii) $\psi(b,z) = g(z)$.
Hint: Divide the problem into two parts.

(9) (b) $\nabla^2 \psi(r,z) = -F(r,z); \quad 0 < r < b,\ 0 < z < \pi.$

(i) $\psi(r,0) = 0$, (ii) $\psi(r,\pi) = f(r)$, (iii) $\psi(b,z) = g(z)$.

(8) 5. (a) The moment of inertia matrix is given by

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{bmatrix}$$

when referred to the xyz coordinate system. Find the principal moments of inertia and unit vectors along the three principal axes.

(3) (b) Use matrix methods to identify and sketch

$$4x_1^2 + 4x_2^2 + 4x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3 = 18.$$

(2) (c) Obtain the matrix of transformation from the (x_1, x_2, x_3) axes to those axes with respect to which the equation has no cross terms.

(1) (d) Find the cosine of the angle of rotation corresponding to the above transformation.

(1) (e) Indicate without performing the explicit computations, how to find the axis of rotation.

(3) (f) Does

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-[4x^2-2xy+4y^2+2xz+4z^2-2yz]}\,dx\,dy\,dz$$

converge? If so, evaluate the integral.

(5) 6. (a) For

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

evaluate e^{At}.

(6) (b) Solve

$$\begin{aligned} \dot{x}_1 &= 4x_1 - x_2 + 3; \quad x_1(0) = -1 \\ \dot{x}_2 &= x_1 + 2x_2 + 3; \quad x_2(0) = 3. \end{aligned}$$

(11) 7. Solve

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-4}{t^2} & \frac{4}{t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t \\ 4 \end{bmatrix}; \ t \geq 1$$

with $X(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Good Luck!

McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATHEMATICS 189-266A

LINEAR ALGEBRA & BOUNDARY VALUE PROBLEMS

Examiner: Professor C. Roth
Associate Examiner: Professor N.G.F. Sancho

Date: Friday, December 19, 1997
Time: 2:00 P.M. - 6:00 P.M.

This exam comprises the cover, 3 pages of questions and 1 page of useful information.