McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATH 255-001

HONOURS ANALYSIS 2

Examiner: Professor K. GowriSankaran
Associate Examiner: Professor S. Drury

Date: Wednesday April 12, 2006
Time: 2:00 pm - 5:00 pm

INSTRUCTIONS

(a) Answer questions in the exam booklets provided.

(b) All questions count equally.

(c). This is a closed book exam. No computers, notes or text books are permitted.

(d) Calculators are not permitted.

(e) Use of a regular and or translation dictionary is not permitted.

This exam comprises of the cover page, and 2 pages of 6 questions.

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MATH 255 FINAL EXAMINATION

No Calculators

Answer all questions. All questions count equally.

1. Decide if the following statements are true or false.

   (a) $\sum_{1}^{\infty} na_n$ converges $\Longrightarrow \sum_{1}^{\infty} a_n$ converges

   (b) $f(x) := \sin(1/x)$ is Riemann - Darboux integrable on $[0, 1]$

   (c) Suppose f is continous on $[1, \infty)$ and $\int_1^\infty f$ is finite then $f(x)$ tends zero as $n \to \infty$.

2. Justify your conclusions.
   (a) $f_n(x) := nx(1-x^2)^n$ for each $n, x \in [0, 1]$. Prove that (f_n) converges pointwise but the convergence is not uniform.

   (b) Suppose (a_n) is a bounded sequence of real numbers such that $\sum_{n=0}^{\infty} a_n$ diverges. Show that $\sum_{0}^{\infty} a_n x^n$ has radius of convergence 1.

3. (a) Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}$ by using the definition of Riemann - Darboux integral of an appropriate continuous function.

   (b) Let $f : [0, 1] \to \mathbb{R}$ be continuous. Prove that the Cauchy-Reimann integral $\int_0^1 \frac{f(x)}{\sqrt{1-x^2}}\, dx$ is finite.

4. (a) Suppose P and Q are polynomial functions of degree p and q respectively. Suppose further that whatever be $k \in \mathbb{N}$, $Q(k) \neq 0$. Prove that $\sum_{k=1}^{\infty}(-1)^{k+1} \frac{P(k)}{Q(k)}$ is convergent if and only if $p < q$.

   (b) Show that $\sum_{1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < 2 + \sqrt{2}$.

5. (a) Let K_1 and K_2 be two non-void compact subsets of $\mathbb{R}$ such that $K_1 \cap K_2 = \emptyset$. Prove that $\inf\{|x-y| : x \in K_1, y \in K_2\}$ is > 0.

(b) Let (f_n) be a sequence of continuous functions on $\mathbb{R}$ such that $\forall\, x \in [0,1]$, $f_n(x) \leq f_{n-1}(x) \quad \forall n \geq 2$. Suppose $\forall x \in [0,1], \ f_n(x) \to 0$. Prove that (f_n) converges to 0 uniformly.
[Hint:$\forall\, \epsilon > 0$, prove that $\{V_m\}$ is an open cover of $[0,1]$ if $V_m = \{x : f_m(x) < \epsilon\}$]

6. (a) Define/Explain the following concepts.

   (i) d is a metric on a set X

   (ii) $x_n \in X$ and $y \in X, x_n \to y$ in the metric d

   (b) Let F be a non-void closed subset of a space X with a metric d and let $\rho(x, F) = \inf\{d(x,y) : y \in F\}$.

   Prove that ρ is a continuous function on X such that $\{x : \rho(x, F) = 0\} = F$.

   (c) Let $F_1 \subset X, \ F_2 \subset X$ be two disjoint non-void closed subsets of the metric space (X, d). Prove that $f(x) = \dfrac{\rho(x, F_1)}{\rho(x, F_1) + \rho(x, F_2)}$ is a continuous function $X \to [0,1]$.

   (d) Use the f in (c) above and show that there are open sets $V_j \supset F_j$ such that $V_1 \cap V_2 = \emptyset$.